Complicated Dynamics in Simple Monetary Models*

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Abstract

This paper analyzes stationary and nonstationary equilibria in search models of monetary exchange with indivisible assets. Two specifications are considered: the usual one, based on bargaining; and a more novel one, with arguably better microfoundations, based on price posting and directed search. For each case, and especially for posting, we generalize assumptions in previous analyses and prove new results. Both models have equilibria where endogenous variables change over time as self-fulfilling prophecies, including sunspot equilibria where they fluctuate randomly. Interestingly, and perhaps surprisingly, we prove there are no equilibria where the variables cycle deterministically over time.

Key words: money, asset prices, dynamics, cycles, sunspots

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1 Introduction

This paper studies stationary and nonstationary equilibria in second-generation (i.e., indivisible-asset) search models of monetary exchange. Two specifications are considered: the usual one, based on bargaining, as in Shi (1995) or Trejos and Wright (1995); and one with arguably better microfoundations, based on directed search and posting, as in Julien et al. (2008). In each case we generalize previous analyses and prove new results. Both models can have multiple steady states, as well as equilibria where endogenous variables change over time as self-fulfilling prophecies, including sunspot equilibria where they fluctuate randomly. This is not too surprising given previous results. Then we prove there are no cyclic or chaotic equilibria where endogenous variables vary deterministically over time. This is surprising given what we know about other models.[1] As monetary theorists, we think it is good to know when cyclic equilibria are possible and when they are not, and these results resolve some previously open questions.

We are interested in second-generation models that restrict individual asset holdings to $m \in \{0, 1\}$ because they allow us to succinctly address key questions: What frictions make monetary exchange an equilibrium or an efficient arrangement? When are assets valued for liquidity? Do liquidity considerations lead to multiplicity or volatility? How does this impinge on allocations and welfare? While some of these issues can be studied with divisible assets, that is complicated due to an endogenous distribution of m across agents, and hence requires special assumptions or numerical methods.[2] Hence, there is still a role for second-generation monetary theory. By analogy, indivisible-asset models are deployed to good effect in studies of middlemen by Rubinstein and Wolinsky (1987), banking by Cavalcanti and Wallace (1999), and

[1] See Azariadis (1993) for a textbook treatment of cycles in different monetary models. In a more closely related specification, Burdett et al. (2016) show that cyclic equilibria exist in a model with "noisy search" in the sense of Burdett and Judd (1983).

[2] Examples of analytic results in monetary search models include Shi (1997), Green and Zhou (1998, 2002), Lagos and Wright (2005), Menzio et al. (2013), and Rocheteau et al. (2015a,b). Numerical work includes Molico (2006) and Chiu and Molico (2010, 2011).

OTC financial markets by Duffie et al. (2005). Similarly, the Pissarides (2000) and Burdett and Coles (1997) models of employment and partnership formation assume a firm can hire n workers and an agent can have n partners, with the restriction $n \in \{0,1\}$, yet they still provide important insights. Indeed, $\{0,1\}$ restrictions have proved useful in search theory going at least since Diamond (1982). So, in pursuing these models, at least we are in good company.

Given these models remain relevant in research and in the classroom, we want to further explore their properties. Here we relax several assumptions in previous analyses and prove more results. Our characterization of existence, multiplicity vs uniqueness, and dynamics extends work on bargaining models by using generalized bargaining solutions and meeting technologies. We also consider assets with any rate of return, not just fiat money. While Trejos and Wright (2016) also allow general assets, they only examine Kalai bargaining. Our solution concept nests Kalai, as well as Nash bargaining, and analogous results for the latter are not available in the literature. Also, while Julien et al. (2008) study price posting and directed search, they use a very special meeting technology, only consider fiat money, and only examine steady state.

The rest of the paper is organized as follows. Section 2 describes the environment. Sections 3 and 4 present results for bargaining and posting, respectively. Section 5 considers extensions, and Section 6 concludes. A few technical results are contained in the Appendix.[3]

[3] See the survey by Lagos et al. (2017) for more on the literature, but here is a synopsis: First-generation monetary models have indivisible assets and indivisible goods (e.g., Kiyotaki and Wright 1989,1993). Second-generation models like those we study make goods divisible to endogenize prices. Third-generation models make goods and assets divisible but, as fn. 2 says, they require special assumptions or numerical methods. As argued by Wallace in Altig and Nosal (2013), there are tradeoffs, and neither second- nor third-generation models dominate. In existing second-generation models, in addition to baragaining, people have considered mechanism design (Wallace and Zhu 2007; Zhu and Wallace 2007), auctions (Julien et al. 2008), and posting with random or noisy search (Curtis and Wright 2004; Burdett et al. 2016); no one has used our generalized bargaining specification. Our meeting process also generalizes the usual one, and complements Coles (1999), Corbae et al. (2003), Matsui and Shimizu (2005) and Goldberg (2007). The only version with directed search and posting is Julien et al. (2008), but again they focus on steady state, while we study dynamics.

2 Environment

A $[0,1]$ continuum of infinitely-lived agents meet bilaterally in discrete time. There is a set of non-storable goods $\mathcal{G}$, where each agent of type i consumes a subset $\mathcal{G}^i$ and produces $g \notin \mathcal{G}^i$, implying potential gains from trade. For symmetry, the measure of every type i is the same. Any $g \in \mathcal{G}^i$ gives type i utility $u(q)$ from q units, and has production cost $c(q)$, where $u(0) = c(0) = 0$ and $u(\bar{q}) = c(\bar{q}) > 0$ for some $\bar{q} > 0$. As usual, $u'(q), c'(q) > 0$, $u''(q) < 0$ and $c''(q) \geq 0$ $\forall q > 0$. We also sometimes impose $u'(0) = \infty$ or $c'(0) = 0$, and let $q^* \in (0, \hat{q})$ be the efficient quantity, $u'(q^*) = c'(q^*)$. Specialization is modeled as follows: when i and j meet, δ is the probability i produces $g \in \mathcal{G}^j$ and j produces $g^j \in \mathcal{G}^i$ – a double coincidence – while σ is the probability i produces $g \in \mathcal{G}^j$ and j produces $g^j \notin \mathcal{G}^i$ – a single coincidence.[4] For simplicity we set $\delta = 0$ to rule out barter. Then, to preclude credit, we assume agents cannot commit and histories are private information (Kochelakota 1998). Now assets have a role in facilitating trade.

In this economy there is one asset that can serve in that capacity, and it has a flow return ρ in terms of utility. If $\rho > 0$ it can be interpreted as a dividend, or fruit from a tree, as in standard asset-pricing theory; if $\rho < 0$ it can be a cost of holding the asset, as in many models of commodity money; and if $\rho = 0$ the asset is fiat currency according to standard usage. While much work has focused on $\rho = 0$, we go beyond that not only for the sake of generality, but because one might argue that pure fiat currency never actually existed (Goldberg 2005). In any case, an individual's asset position is $m \in \{0, 1\}$. With a fixed supply of assets, $M \in (0, 1)$, at any point in time a measure M of agents, called buyers, have $m = 1$, while a measure $1 - M$, called sellers, have $m = 0$. After trade a buyer becomes a seller and vice versa.

[4] A common specification (e.g., Aiyagari and Wallace 1991) has K goods and K types of agents, with type k consuming good k and producing good $k + 1$ mod K. In a random meeting, $K = 2$ implies $\delta = 1/2$ and $\sigma = 0$, while $K > 2$ implies $\delta = 0$ and $\sigma = 1/K$. Alternative formulations with σ and δ endogenous can be found in Kiyotaki and Wright (1991,1993).

A defining feature of these models is that agents trade with each other – not merely with their budget lines – which means one must specify the meeting process. Consider a set of buyers with measure n_1 and sellers with measure n_0, and let $b = n_1/n_0$ be the buyer/seller ratio, or market tightness. As standard, following Pissarides (2000), an abstract technology yields the number of buyer-seller meetings as a function $\mu = \mu(n_1, n_0)$. In each period of discrete time, a seller's probability of meeting a buyer is $\alpha_0 = \mu(n_1, n_0)/n_0$, and a buyer's probability of meeting a seller is $\alpha_1 = \mu(n_1, n_0)/n_1$. Given $\mu(n_1, n_0) \leq \min(n_1, n_0)$, we have $\alpha_0, \alpha_1 \leq 1$. It is standard to also assume μ displays constant returns, so that $\alpha_0 = \mu(b, 1) \equiv \alpha(b)$ and $\alpha_1 = \alpha(b)/b$, where $\alpha(b)$ is increasing in b, $\alpha(b)/b$ is decreasing, $\alpha''(b) < 0$, and $\alpha(0) = 0$.

The typical specification in monetary economics following Kiyotaki and Wright (1991) has $\mu = An_1n_0/(n_1 + n_0)$. This implies $\alpha_0 = An_1/(n_1 + n_0) = Ab/(1 + b)$ and $\alpha_1 = An_0/(n_1 + n_0) = A/(1 + b)$, with the interpretation that an agent meets someone with probability $A \leq 1$, and each meeting is a random draw from the population. Then $n_1 = M$ and $n_0 = 1 - M$ implies $\alpha_0 = AM$, $\alpha_1 = A(1 - M)$ and $\alpha_0 + \alpha_1 = A \leq 1$, which is far too restrictive for our purposes (see below). An alternative formulation is Julien et al. (2008), who use the urn-ball meeting technology $\mu = An_0 [1 - \exp(-b)]$, motivated by the microfoundations in Julien et al. (2000) and Burdett et al. (2001). Another natural example is $\mu = A \min\{n_1, n_0\}$, where $\alpha_0 = A \min\{b, 1\}$ and $\alpha_1 = A \min\{b, 1\}/b$, which says everyone on the short side of the market has a meeting. In most of what follows below we use a general technology μ.

What matters is not just meeting but matching – i.e., meeting the right type. With random search, given specialization, that probability is $\alpha_0\sigma$ for sellers and $\alpha_1\sigma$ for buyers. With directed search, σ is irrelevant because agents can target appropriate counterparties – e.g., consumers of type i can direct their search toward producers of $g \in \mathcal{G}^i$ – but whether they meet one is still randomly determined by μ. To cleanly compare economies with bargaining and with posting, buyers here can always direct their search to the appropriate type of seller, so the only difference is price determination.[5] Whether agents post or bargain, the value functions for buyers and sellers at t are denoted V_{1t} and V_{0t}, and $\Delta_t \equiv V_{1t} - V_{0t}$ is the capital gain from acquiring the asset.

3 Bargaining

The standard discrete-time Bellman equations are

$$V_{1t} = \alpha_1 \left[u\left(q_t\right) + \beta V_{0t+1} \right] + \left(1 - \alpha_1\right) \beta V_{1t+1} + \rho \quad (1)$$

$$V_{0t} = \alpha_0 \left[\beta V_{1t+1} - c\left(q_t\right) \right] + \left(1 - \alpha_0\right) \beta V_{0t+1}, \quad (2)$$

where $\beta = 1/\left(1+r\right)$ with $r > 0$. Note that in equilibrium $\alpha_1 = \alpha\left(b\right)/b$ and $\alpha_0 = \alpha\left(b\right)$ are pinned down by $b = M/\left(1-M\right)$, but we can conceptualize payoffs for any b. In words (1) says this: with probability α_1 a buyer meets a seller of his good, in which case he consumes q_t and continues without the asset; with probability $1-\alpha_1$ he does not meet such a seller, in which case he continues with the asset; and in both events he enjoys ρ assuming the asset is traded ex dividend. The interpretation of (2) is similar.

For now the terms of trade are determined by bargaining. We use a generic bargaining solution $\beta\Delta_{t+1} = v\left(q_t\right)$, where $v\left(0\right) = 0$, $v'\left(q\right) > 0$, and $c\left(q\right) \leq v\left(q\right) \leq u\left(q\right)$. This gives output as a function of the capital gain from getting the asset and nests many common solution concepts. The standard generalization of Nash (1950) bargaining, e.g., is

$$\max_{q_t} \left[u\left(q_t\right) - \beta\Delta_{t+1} \right]^{\theta} \left[-c\left(q_t\right) + \beta\Delta_{t+1} \right]^{(1-\theta)},$$

[5] Sometimes posting is interpreted as sellers committing to the terms of trade prior to meeting buyers. However, even without commitment, where posting is cheap talk, it can still affect outcomes (Menzio 2007). To avoid this we can interpret the bargaining environment as one where agents cannot communicate – as opposed to cannot commit – prior to meeting. Also, to be clear, agents know where to find the right type of counterparty, but not a particular individual, to preclude credit based on long-term relationships as in Corbae and Ritter (2004).

where θ is buyers' bargaining power. Taking the FOC and rearranging, we get $\beta\Delta_{t+1} = v(q_t)$ where

$$v(q_t) = \frac{\theta u'(q_t)\, c(q_t) + (1-\theta)\, c'(q_t)\, u(q_t)}{\theta u'(q_t) + (1-\theta)\, c'(q_t)}. \tag{3}$$

Similarly, Kalai's (1977) proportional bargaining solution can be written $\beta\Delta_{t+1} = v(q_t)$ where now[6]

$$v(q_t) = \theta c(q_t) + (1-\theta)\, u(q_t). \tag{4}$$

To determine Δ_t, begin by subtracting (1)-(2) to get

$$\Delta_t = \alpha_1 u(q_t) + \alpha_0 c(q_t) + \rho + (1-\alpha_1-\alpha_0)\,\beta\Delta_{t+1}. \tag{5}$$

Given $v'(q_t) > 0$, we can write $q_t = Q(\beta\Delta_{t+1})$ where $Q'(\beta\Delta_{t+1}) = 1/v'(q_t) > 0$. Then (5) defines a difference equation $\Delta_t = \Phi(\Delta_{t+1})$, where

$$\Phi(\Delta_{t+1}) \equiv \alpha_1 u \circ Q(\beta\Delta_{t+1}) + \alpha_0 c \circ Q(\beta\Delta_{t+1}) + \rho + (1-\alpha_1-\alpha_0)\,\beta\Delta_{t+1}, \tag{6}$$

and for any functions f and g, we use $f \circ g(x) \equiv f[g(x)]$ for the composite. In steady state (5) implies

$$\beta\Delta = \frac{\alpha_1 u(q) + \alpha_0 c(q) + \rho}{r + \alpha_1 + \alpha_0}, \tag{7}$$

with the RHS a weighted average of buyers' utility $u(q)$, sellers' cost $c(q)$, and the payoff to keeping the asset as a store of value ρ/r.

To first consider stationary outcomes, combine (7) with $\beta\Delta = v(q)$ to get $\rho = e(q)$, where

$$e(q) \equiv (r + \alpha_1 + \alpha_0)\, v(q) - \alpha_1 u(q) - \alpha_0 c(q). \tag{8}$$

[6]Kalai's solution maximizes a buyer's surplus subject to him getting a share θ of the total surplus (this is not the definition, it is a result implied by his axioms, like maximizing the product of the surpluses is a result implied by Nash's axioms). Unless $u(q) = c(q) = q$ or $\theta \in \{0, 1\}$, Nash and Kalai differ, with the latter having some advantages in monetary theory (Aruoba et al. 2007). Zhu (2016) also shows how the outcome of simple stragetic bargaining can be written $\beta\Delta_{t+1} = v(q_t)$. We adopt the generic formulation $v(q)$ from Gu and Wright (2016), where its properties are derived from simple axioms.

A stationary monetary equilibrium, or SME, is now defined by $\beta\Delta^s = v(q^s)$ where $q^s \in (0, \bar{q}]$ solves $\rho = e(q^s)$, and $q^s \leq \bar{q}$ is needed for $c(q) \leq u(q)$ (i.e., for voluntary trade). Equivalently, SME can be described by $\Delta^s \in (0, \bar{\Delta}]$, where $\bar{\Delta}$ solves $\beta\bar{\Delta} = v(\bar{q})$. A dynamic monetary equilibrium, or DME, is a nonconstant solution to (6) with $\Delta_t \in (0, \bar{\Delta}]$ $\forall t$, or a solution to (5) after eliminating $\beta\Delta_{t+1} = v(q_t)$ with $q_t \in (0, \bar{q}]$ $\forall t$. We call these monetary equilibria, even if the asset is not fiat currency unless $\rho = 0$, because the asset *circulates* as a *medium of exchange*. While we focus on such outcomes, if $\rho \leq 0$ there is always a nonmonetary equilibrium where agents dispose of their assets. Also, if $\rho > 0$ is big agents may prefer to hoard rather than trade assets, although that depends on indivisibility, as discussed in Section 5.

For SME, i.e. for solutions to $\rho = e(q^s)$, previous analyses typically impose a bargaining solution – e.g., Shi (1995) or Trejos and Wright (1995) use symmetric Nash, or a strategic game that amounts to the same thing; Rupert et al. (2001) use generalized Nash; and Trejos and Wright (2016) use Kalai. We start with Kalai to facilitate comparison with Trejos and Wright (2016), and consider other bargaining solutions, plus posting, below. For Kalai, the following Lemma can be stated without proof since it follows from direct calculation.

Lemma 1 *Assume Kalai bargaining, and define $\underline{\theta} \in (0,1)$ and $\bar{\theta} \in (\underline{\theta}, 1)$ by*

$$\bar{\theta} \equiv \frac{r + \alpha_0}{r + \alpha_1 + \alpha_0} \text{ and } \underline{\theta} \equiv \frac{\alpha_0}{r + \alpha_1 + \alpha_0}. \tag{9}$$

Then (1) $\theta > \bar{\theta} \Rightarrow e'' > 0$; (2) $\underline{\theta} < \theta < \bar{\theta} \Rightarrow e' > 0$; and (3) $\theta < \underline{\theta} \Rightarrow e'' < 0$. Also, $e(0) = 0 < e(\bar{q}) = ru(\bar{q})$.

Figure 1 shows $e(q)$ for the three cases in Lemma 1 using this notation: $\bar{\rho} \equiv e(\bar{q})$; $\hat{\rho} \equiv \max_{[0,\bar{q}]} e(q)$ when $\theta < \underline{\theta}$; and $\underline{\rho} \equiv \min_{[0,\bar{q}]} e(q) < 0$ when $\theta > \bar{\theta}$. The left panel shows $\hat{\rho} = \bar{\rho}$, the right shows $\hat{\rho} > \bar{\rho}$, and both are possible. From Figure 1 the set of solutions $q^s \in (0, \bar{q})$ to $e(q^s) = \rho$ is immediate:[7]

[7] This is a slight extension of Trejos and Wright (2016) to a more general meeting technology, and while that does not make the proof much harder, we discuss below why it is interesting.

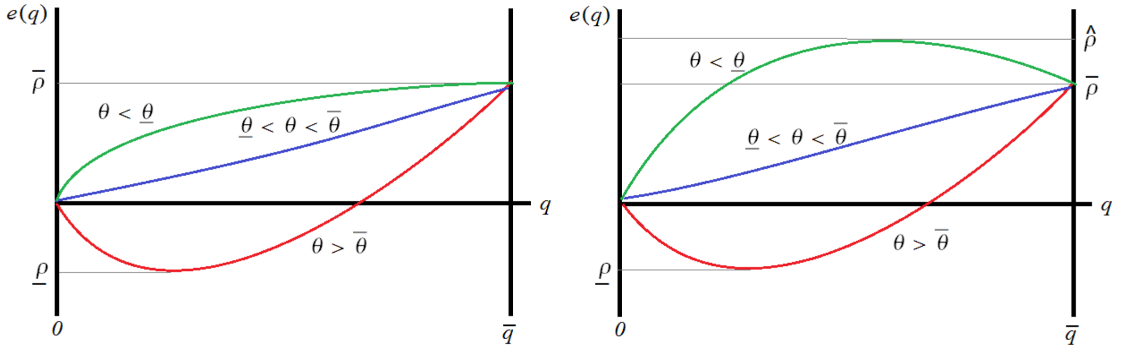

Figure 1: SME with Kalai bargaining

Proposition 1 *Assume Kalai bargaining. (1) $\theta > \bar{\theta}$. If $\rho < \underline{\rho}$ there is no SME; if $\underline{\rho} < \rho < 0$ there are two SME; if $0 \leq \rho < \bar{\rho}$ there is one SME; if $\rho > \bar{\rho}$ there is no SME. (2) $\underline{\theta} < \theta < \bar{\theta}$. If $\rho \leq 0$ there is no SME; if $0 < \rho < \bar{\rho}$ there is one SME; if $\rho > \bar{\rho}$ there is no SME. (3) $\theta < \underline{\theta}$. If $\rho \leq 0$ there is no SME; if $0 < \rho \leq \bar{\rho}$ there is one SME; if $\rho > \bar{\rho}$ there are two subcases: (3a) if $\hat{\rho} = \bar{\rho}$ (as in the left panel of Figure 1) there is no SME, and (3b) if $\hat{\rho} > \bar{\rho}$ (as in the right panel) there are two SME if $\bar{\rho} < \rho < \hat{\rho}$ and no SME if $\rho > \hat{\rho}$.*

Remark 1 *This kind of multiplicity is standard in monetary models, because the value of liquidity is in part a self-referential concept (see Lagos et al. 2017 for more discussion), but here we get precise results on the number of SME and when they exist. Note that we typically ignore nongeneric values for ρ except when they are especially relevant, e.g., $\rho = 0$. Also, one can easily use Figure 1 for comparative statics – e.g., raising θ rotates $e(q)$ clockwise, lowering $\underline{\rho}$ and $\bar{\rho}$, and reducing q whenever SME is unique.*

In terms of stability, the following results are obvious from Figure 2, showing (6) in (Δ_{t+1}, Δ_t) space for different θ and ρ, with stable steady states indicated by green dots and unstable ones by red triangles:[8]

[8]This is not quite a generalization of Trejos and Wright (2016), which is a continuous-time model, but gives analogous discrete-time results for more general meeting technologies.

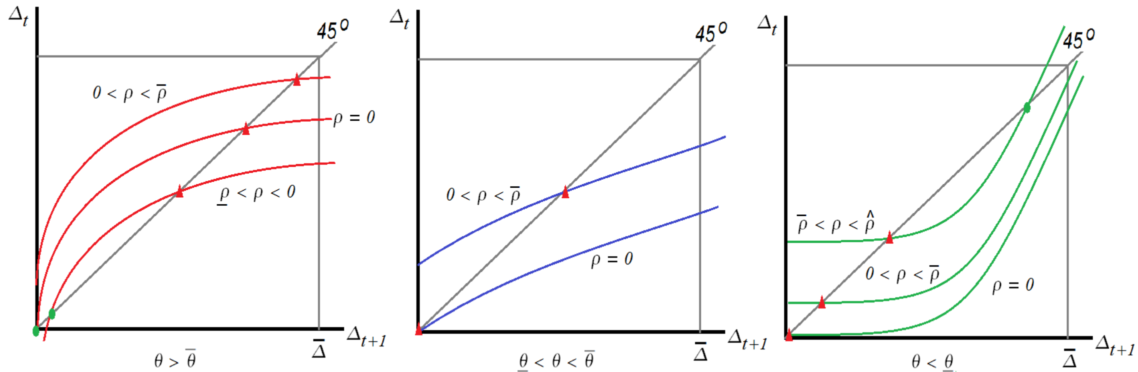

Figure 2: DME with Kalai bargaining

Lemma 2 *Assume Kalai bargaining. (1) $\theta > \bar{\theta}$. If $\underline{\rho} < \rho < 0$ the high SME is unstable and the low one is stable; if $\rho = 0$ the unique SME is unstable and $\Delta = 0$ is stable; and if $0 < \rho < \bar{\rho}$ the unique SME is unstable. (2) $\underline{\theta} < \theta < \bar{\theta}$. The unique SME is unstable. (3) $\theta < \underline{\theta}$. If $0 < \rho \leq \bar{\rho}$ the unique SME is unstable; and if $\bar{\rho} < \rho < \hat{\rho}$ the low SME is unstable while the high one is stable.*

Based on this, the following is immediate:

Proposition 2 *Assume Kalai bargaining. If there are no SME then there are no DME. Otherwise: (1) $\theta > \bar{\theta}$. If $\underline{\rho} < \rho < 0$ there are DME with Δ_t approaching the low SME for any initial Δ_0 in an interval; if $\rho = 0$ there are DME with Δ_t approaching 0 for any Δ_0 in an interval; and if $0 < \rho < \bar{\rho}$ there are no DME. (2) $\underline{\theta} < \theta < \bar{\theta}$. There are no DME. (3) $\theta < \underline{\theta}$. If $\bar{\rho} < \rho < \hat{\rho}$ there are DME with Δ_t approaching the high SME for any Δ_0 in an interval.*

In DME, Δ_t and q_t vary over time as a self-fulfilling prophecy. This cannot happen in the middle panel of in Figure 2, where there is a unique SME, it is unstable, and any path other than the steady state eventually violates $\Delta_t \in [0, \bar{\Delta}]$. It can happen when there is a stable SME, in which case a path leading to it from any initial Δ_0 is a DME. The basin of attraction for a stable SME can be large: in the left panel, e.g., if $\underline{\rho} < \rho < 0$ then for any initial Δ_0 between the origin and the high SME there is a path converging to the low SME.

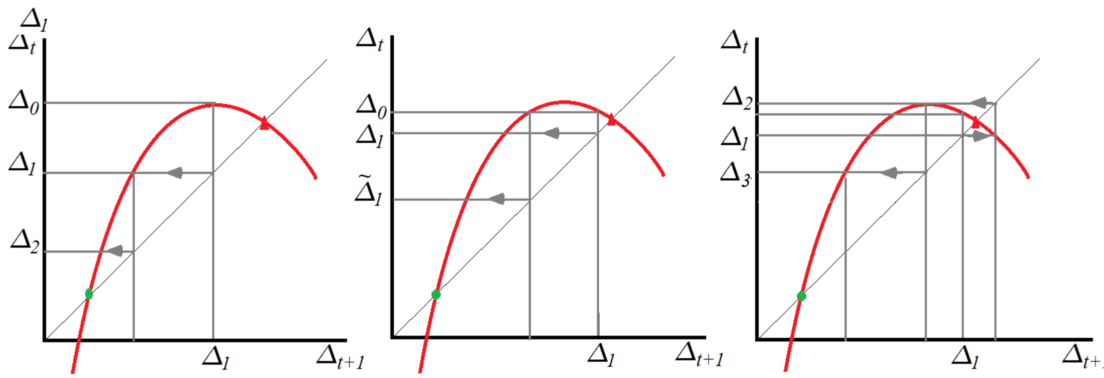

Figure 3: Interesting outcomes when $\Phi'(\Delta) < 0$

Notice that $\Phi'(\Delta^s) < 0$ admits several interesting economic phenomena. First, there are DME starting from an initial Δ_0 above the high SME, as shown in the left panel of Figure 3. Second, for a given Δ_0 there can be multiple DME, as seen in the middle panel. Third, there are nonmonotone DME, oscillating around the high SME before converging to the low SME, as seen in the right panel. Based on this, clearly, it is relevant to allow $\Phi'(\Delta) < 0$, and we verify below that this is possible with our formulation but *not* with the special meeting technology standard in the previous literature.

Before proceeding, one might conjecture that there can be limit cycles around a steady state. We now show this is *not* the case, and the proof allows *any* bargaining solution of the form $\beta\Delta_{t+1} = v(q_t)$.

Proposition 3 *Deterministic cyclic equilibria do not exist for any $v(q)$.*

Proof: First, $u', c', Q' > 0$ and $\alpha_0, \alpha_1 \in (0, 1]$ imply

$$\begin{aligned} \Phi'(\Delta_{t+1}) &= \alpha_1 \beta u' Q' + \alpha_0 \beta c' Q' + (1 - \alpha_1 - \alpha_0)\beta \\ &> (1 - \alpha_1 - \alpha_0)\beta > -1. \end{aligned}$$

If $\Phi'(\Delta) > 0$ the dynamics are monotone and there are no cyclic equilibria. That is the case when $\alpha_0 = AM$ and $\alpha_1 = A(1 - M)$, since then $\alpha_1 + \alpha_0 = A \leq 1$, but $\Phi'(\Delta) > 0$ is not true in general. Still, the result $\Phi'(\Delta) > -1$ is enough to rule out two-period cycles. To see this, note from (6) that $\Phi'(0) > 0$, so $\Phi'(\Delta^s) > -1$ implies Φ and Φ^{-1} cannot cross off the 45^o line (draw the picture). Hence there are no cycles of period 2. Since the Sharkovskii Theorem (see Azariadis 1993) says that cycles of period $n > 2$ imply cycles of period 2, there do not exit cycles of any period here. ■

While Proposition 3 uses for a general $v(q)$, Propositions 1-2 use Kalai bargaining, but they can be generalized as follows. In terms of SME, from (8) we still have $e(0) = 0 < e(\bar{q})$ for any $v(q)$, which implies existence of SME when $\rho \in (0, \hat{\rho})$ and nonexistence when $\rho > \hat{\rho}$. However, for $\rho \in (0, \bar{\rho})$ we cannot say that there is exactly one SME, only generically an odd number, and for $\rho \in (\bar{\rho}, \hat{\rho})$ we cannot say that there are exactly two SME, only generically an even number. In terms of Figure 1, the situation for a general $v(q)$ looks similar, except $e(q)$ is not necessarily convex or concave, and might "wiggle" around. That takes care of $\rho \geq 0$. For $\rho \leq 0$ we need an additional assumption. To proceed, first note that

$$e'(q) = (r + \alpha_1 + \alpha_0) v'(q) - \alpha_1 u'(q) - \alpha_0 c'(q).$$

Assuming $e'(0) < 0$, we have existence of SME when $\rho \in (\underline{\rho}, 0)$, and not when $\rho < \underline{\rho}$, but again we cannot say there are exactly two SME when they exist.[9]

The situation is similar for DME, where $\Phi(\Delta)$ can "wiggle" in the analog to Figure 2 for a general $v(q)$. As usual, if there are many steady states, generically they alternate between stable and unstable. Hence, we know quite a lot about both SME and DME, although not the exact number of equilibria, with general bargaining. The next step is to derive similar results with posting.

[9] To guarantee $e'(0) < 0$ one requires some condition on $v'(0)$, of course. A simple such condition is $(r + \alpha_1 + \alpha_0) v'(0) < \alpha_1 u'(0)$, which does rule out some bargaining solutions – e.g., as should be obvious, there is no SME when $\rho \leq 0$ and $v(q) = u(q)$, which means take-it-or-leave-it offers by sellers. This should be obvious because that bargaining solution gives buyers no surplus, and hence agents are not willing to produce to get assets for which $\rho \leq 0$.

4 Posting

Posting with directed search, also called competitive search, has the terms of trade designed by agents to attract trading partners: in addition to consumers of good g knowing where to find the right producers, for any g, the market can further segment into submarkets identified by pairs (q, b), where agents commit to trade q for the asset and b is tightness.[10] Whether buyers and sellers meet is still random, with $\alpha_0 = \alpha(b)$ and $\alpha_1 = \alpha(b)/b$, except b is now tightness in a particular submarket. In equilibrium it turns out that $b = M/(1 - M)$, but in principle b can vary across submarkets.

As discussed in the survey by Wright et al. (2016), competitive search has several attractive properties, and in particular has arguably better microfoundations than bargaining for dynamic modeling. To explain this, note that one can always impose a cooperative equilibrium concept, like Nash or Kalai bargaining, but one might worry about strategic foundations. Consider a standard extensive-form game (e.g., Rubinstein 1982) where a buyer and seller in a stationary setting make counteroffers of q until one is accepted, with the time between offers denoted $\eta > 0$. Binmore et al. (1986) show the unique subgame-perfect equilibrium has the first offer accepted, and labeling it q^η to indicate dependence on timing, $q^\eta \to q^N$ as $\eta \to 0$, where q^N is the generalized Nash solution. Similarly, Dutta (2012) provides strategic foundations for Kalai bargaining. These results are commonly regarded as support for cooperative bargaining theory.[11]

[10] A submarket is defined by the set of sellers posting (q, b) and the set of buyers directing their search toward them. While we think of sellers posting (q, b) to attract buyers, under certain conditions the outcome is the same if buyers post (q, b) to attract sellers, or third parties called market makers post (q, b) to attract buyers and sellers to their submarkets (see, e.g., Delacroix and Shi 2016). Also, it does not matter if b is posted or agents simply figure it out given what others are doing.

[11] The quest for the strategic foundations of axiomatic solutions is dubbed the "Nash program" by Binmore (1987). As Serrano (2005) says, "Similar to the microfoundations of macroeconomics, which aim to bring closer the two branches of economic theory, the Nash program is an attempt to bridge the gap between the two counterparts of game theory (cooperative and non-cooperative). This is accomplished by investigating non-cooperative procedures that yield cooperative solutions as their equilibrium outcomes."

As discussed in Coles and Wright (1998), Ennis (2001) and Coles and Muthoo (2003), the results in Binmore et al. (1986) hold in dynamic equilibrium settings like the one under consideration *if* attention is restricted to steady states, but *not* otherwise. When $\eta \to 0$, in fact, subgame-perfect equilibrium entails a path for q satisfying a differential equation with q^N as a steady state, but $q \neq q^N$ out of steady state except in special cases. One such case is bargaining with $\theta = 1$ or $\theta = 0$, and another is $u(q) = c(q) = q$, but these are far too restrictive for our purposes. The above papers also argue that using Nash out of steady state is tantamount to using the extensive-form game with myopic agents, who negotiate as if economic conditions were constant when they are not. Moreover, it matters for results: with forward-looking strategic bargaining, e.g., Coles and Wright (1998) construct dynamic equilibria that are not possible with Nash bargaining. However, forward-looking strategic bargaining is complicated in this model. An advantage of posting is that the analysis is easy even with strategic forward-looking agents.

The value functions still satisfy (1)-(2), Δ_t satisfies (5), and steady state satisfies (7), but now, to determine (q, b) we maximize sellers' expected payoff subject to buyers' getting their expected market payoff, which is determined in equilibrium but taken as given by individuals. Formally, the problem is

$$V_{0t} = \max_{(q_t, b_t)} \{\alpha(b_t)[\beta\Delta_{t+1} - c(q_t)] + \beta V_{0t+1}\} \tag{10}$$

$$\text{st } \frac{\alpha(b_t)}{b_t}[u(q_t) - \beta\Delta_{t+1}] + \rho + \beta V_{1t+1} = V_{1t}. \tag{11}$$

We show in the Appendix that the SOC's hold at any interior solution to the FOC's, so there is a unique such solution.[12] This means all open submarkets are the same; or, equivalently, given a CRS meeting technology, there is just one submarket. Moreover, the FOC's in the Appendix imply

$$\beta\Delta_{t+1} = \frac{\varepsilon(b_t)\, u'(q_t)\, c(q_t) + [1 - \varepsilon(b_t)]\, c'(q_t)\, u(q_t)}{\varepsilon(b_t)\, u'(q_t) + [1 - \varepsilon(b_t)]\, c'(q_t)} \tag{12}$$

[12] This result is an independent contribution, since it easily generalizes to other applications of competive search, where the SOC's are often ignored (Wright 2017).

where $\varepsilon \equiv b\alpha'(b)/\alpha(b)$ is the elasticity of $\alpha(b)$, which in equilibrium is pinned down by $b = M/(1-M)$.

Consistent with other applications of competitive search equilibrium, (12) is identical to what one gets with generalized Nash bargaining, except ε replaces bargaining power θ (sometimes this is described by saying competitive search delivers endogenously the efficiency condition in Hosios 1990). In the popular special case $\alpha_0 = M$ and $\alpha_1 = 1-M$, it is easy to check $\varepsilon = 1-M$, and q looks like it is determined by generalized Nash with buyers' bargaining power set to the probability they meet sellers, $\theta = 1-M$. In any case, this equivalence between directed search and generalized Nash bargaining is useful because any results we prove for the former also hold for the latter.

As in Section 3, write (12) as $\beta\Delta_{t+1} = v(q_t)$ and invert it to get $q_t = Q(\beta\Delta_{t+1})$, then substitute this into (5) to get $\Delta_t = \Phi(\Delta_{t+1})$, just like (6). However, the method here is different, because the $v(q_t)$ implied by (12) is more complicated. To proceed, first, equate (7) to (12) and rearrange to get

$$\frac{(1-\varepsilon)\,c'(q)}{\varepsilon u'(q)} = \frac{\alpha_1 u(q) - (r+\alpha_1)\,c(q) + \rho}{(r+\alpha_0)u(q) - \alpha_0 c(q) - \rho}. \tag{13}$$

Call the LHS $L(q)$ and the RHS $R(q)$. The properties of $L(q)$ are simple and hence stated without proof:

Lemma 3 *$L(0) = 0$ and $L'(q) > 0$ $\forall q \in (0, \bar{q})$.*

The properties of $R(q)$ are harder to describe, as there are various cases depending on ρ, and we have to worry about the denominator in (13) hitting 0. Let us first dispense with $\rho = \bar{\rho}$, in which case $q = \bar{q}$ is a SME. Now consider $\rho \neq \bar{\rho}$. It is easy to see $R(\bar{q}) = -1$, $R(0) = -1$ if $\rho \neq 0$ and $R(0) = \alpha_1/(r+\alpha_0)$ if $\rho = 0$. For other properties, the somewhat difficult proof of the next Lemma is in the Appendix, but the results should be clear from Figure 4.

Lemma 4 *(a) As shown in the upper left panel of Figure 4, $\rho = 0$ implies $R'(q) < 0$ $\forall q$. (b) As shown in the upper right panel, $\rho < 0$ implies there exists $\tilde{q} \in (0, \bar{q})$ such*

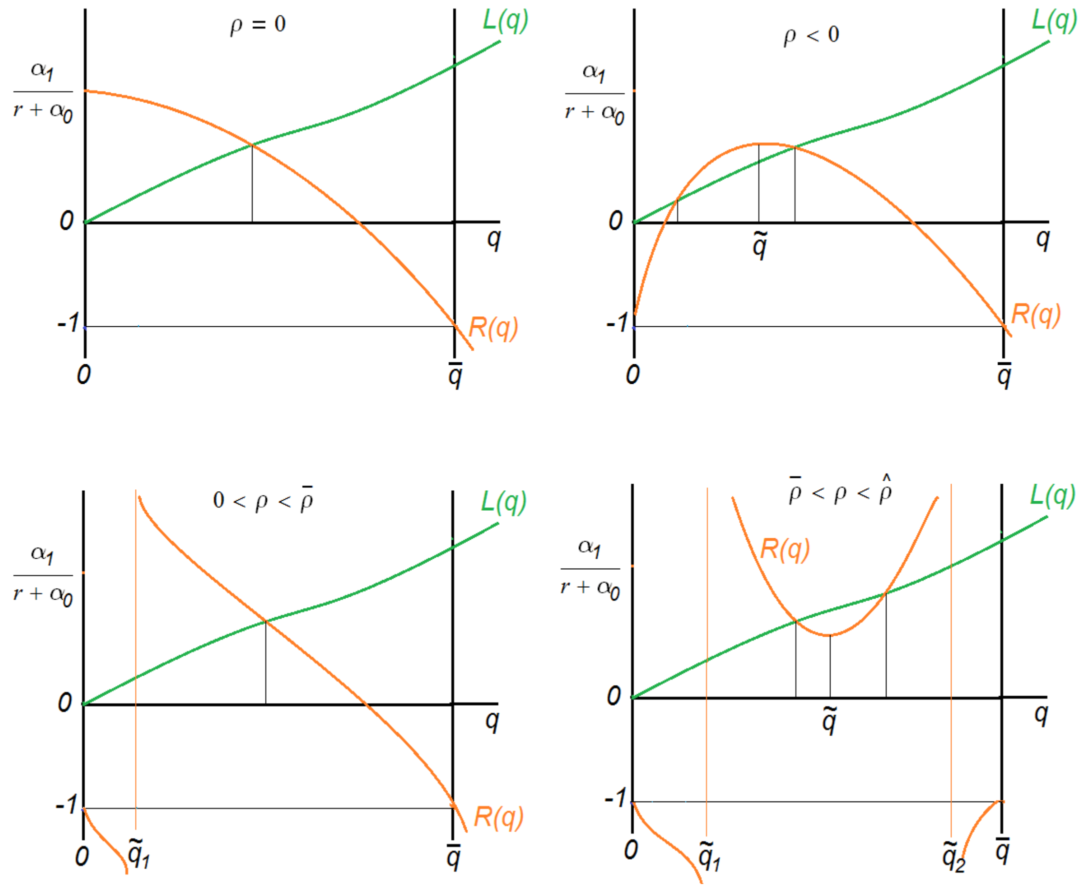

Figure 4: Four possible cases for the function $R(q)$

that $R'(q) > 0\ \forall q < \tilde{q}$ *and* $R'(q) < 0\ \forall q > \tilde{q}$. *(c) As shown in the lower left,* $0 < \rho < \bar{\rho}$ *implies there exists* $\tilde{q}_1 \in (0, \bar{q})$ *such that* $R(q) < 0\ \forall q < \tilde{q}_1$. *Also* $R(q) > 0 \Rightarrow R'(q) < 0\ \forall q > \tilde{q}_1$. *Also,* $R(q) \nearrow \infty$ *as* $q \searrow \tilde{q}_1$. *(d) As shown in the lower right,* $\bar{\rho} < \rho < \hat{\rho}$ *implies there exist* $\tilde{q} \in (0, \bar{q})$, $\tilde{q}_1 \in (0, \tilde{q})$ *and* $\tilde{q}_2 \in (\tilde{q}, \bar{q})$ *such that* $R(q) > 0\ \forall q \in (\tilde{q}_1, \tilde{q}_2)$, $R(q) < 0\ \forall q \notin (\tilde{q}_1, \tilde{q}_2)$, $R'(q) < 0\ \forall q < \tilde{q}$ *and* $R'(q) > 0\ \forall q > \tilde{q}$. *Also,* $R(q) \nearrow \infty$ *as* $q \searrow \tilde{q}_1$ *or* $q \nearrow \tilde{q}_2$.

Lemma 4 immediately yields the following result:

Proposition 4 *Assume posting. For* $\rho \in [0, \bar{\rho})$, *SME exists uniquely. For* $\rho < 0$, *multiple SME exist if* $|\rho|$ *is not too big and no SME exist if* $|\rho|$ *is too big. For* $\rho > \bar{\rho}$, *multiple SME exist if* ρ *is not too big and no SME exist if* ρ *is too big.*

Proof: A SME is a solution to $R(q) = L(q)$ with $q \in (0, \bar{q})$. In the top left panel of Figure 4, with $\rho = 0$, SME exists because $R(0) > L(0)$ and $R(\bar{q}) < L(\bar{q})$, and it is unique because $R'(q) < 0$. In the bottom left panel, with $\rho \in (0, \bar{\rho})$, there is a critical point $\tilde{q}_1 \in (0, \bar{q})$, and SME exists at $q \in (\tilde{q}_1, \bar{q})$ because $R(q) > L(q)$ for q near $\tilde{q}_1$ and $R(\bar{q}) < L(\bar{q})$. It is again unique because $R'(q) < 0$ when $R(q) > 0$. In the bottom right, with $\rho \in (\bar{\rho}, \hat{\rho})$ there are two critical points $\tilde{q}_1$ and $\tilde{q}_2$, and we have multiple SME when ρ is not too big, as shown. However, when ρ is too big, $R(q)$ shifts up too much and there is no SME. Finally, in the top right panel, with $\rho < 0$, there are multiple SME when $|\rho|$ is not too big. However, if $|\rho|$ is too big $R(q)$ shifts down too much and there is no SME. ■

Remark 2 *The method of describing SME in terms of $L(q) = R(q)$ also applies to Kalai bargaining: simply replace the LHS in (13) with $L(q) = (1 - \theta)/\theta$. Since the $L(q)$ curve is now a horizontal line we get stronger results with Kalai – e.g., exactly two SME when there is multiplicity. Still, the proof of Proposition 1 above is easier since it does not rely on properties of $R(q)$ in Lemma 4.*

Figure 5 shows the dynamics, where again stable and unstable steady states are indicated by green dots and red triangles. The analog to Proposition 2 is this:

Proposition 5 *Assume posting. If there are no SME then there are no DME. Otherwise: (a) For $\rho = 0$ there are DME with Δ_t approaching 0 from any Δ_0 in an interval. (b) For $\rho < 0$ there are DME with Δ_t approaching every stable SME from any Δ_0 in an interval. (c) For $\rho \in (0, \bar{\rho})$ there are no DME. (d) For $\rho \in (\bar{\rho}, \hat{\rho})$ there are DME with Δ_t approaching every SME for any Δ_0 in an interval.*

Remark 3 *To clarify, when we say there are DME with Δ_t approaching a stable SME, this mean the following: if there are exactly two SME the stable one can be either the low or the high SME, as in the upper right or lower right of Figure 5; and when there are more than two SME, generically they alternate between stable and unstable.*

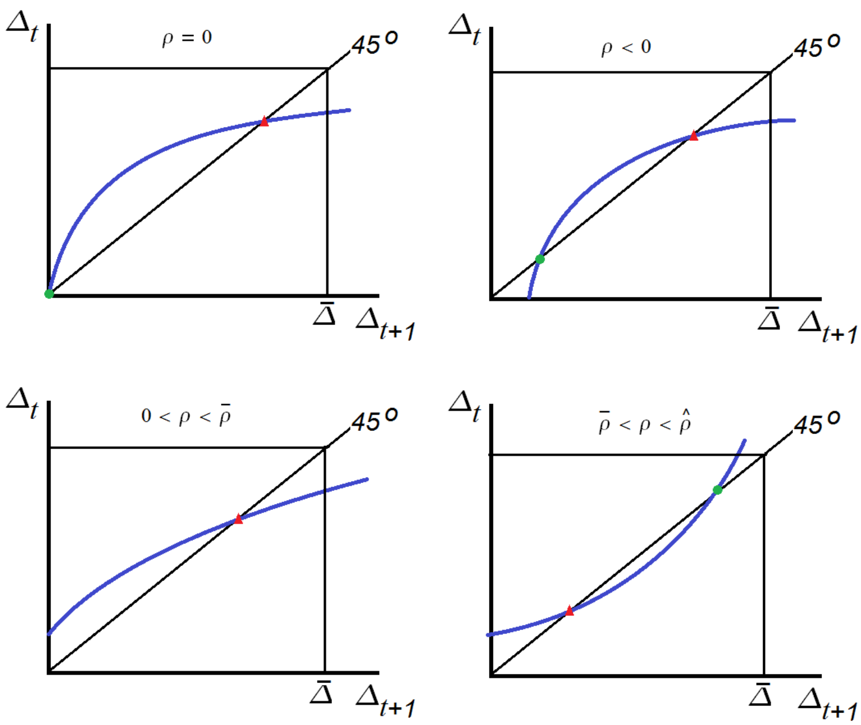

Figure 5: DME with directed search and posting

We now exploit the formal equivalence between posting and generalized Nash bargaining – the latter is the same as the former except ε replaces θ – to significantly generalize existing results.[13]

Proposition 6 *Assume generalized Nash bargaining. Then SME and DME are exactly as described in Propositions 4 and 5.*

The next result is that there are no cycles with posting, something not previously known. Here it follows immediately from Proposition 3, since that applies to any bargaining solution $v(q)$, including Nash, which is formally the same as posting.

Proposition 7 *Deterministic cyclic equilibria do not exist with posting.*

[13] Most of the studies with Nash bargaining cited above use the special case $\alpha_0 = M$, $\alpha_1 = 1 - M$ and $\rho = 0$, and usually focus on steady state; Trejos and Wright (1995) discuss some dynamics, and Shi (1995) discusses sunspots, but only for the above special case, plus $\theta = 1/2$.

We worked out several numerical examples using utility function

$$u(q) = \frac{(q+\varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}}{1-\gamma},$$

and cost function $c(q) = q$. We used two common specification for the meeting technology, $\alpha(b) = 1 - e^{-b}$ and $\alpha(b) = \min\{1, b\}$ on the right. With $\gamma = 5$, $\varepsilon = 0.4$, $\beta = 0.9$, Nash bargaining with $\theta = 0.5$, tightness $b = 2$, and $\rho = 0$, e.g., it is easy to check $\Phi'(\Delta) < 0$ in SME. This is important because Propositions 3 and 7 are virtually trivial with $\Phi'(\Delta) > 0$, and the examples show the extra effort is worth it because that is indeed possible.

5 Extensions

5.1 Sunspots

Although deterministic cycles are impossible, there can be sunspot equilibria where Δ_t and q_t fluctuate randomly even while fundamentals are constant. This has been shown previously in related models by Shi (1995), Ennis (2001) and Trejos and Wright (2016), although they use continuous time, and have other differences discussed above. For completeness, we proceed here in discrete time, with a general meeting technology $\alpha(b)$ and trading protocol $v(q)$.

Consider a random variable $S \in \{A, B\}$, where S switches to $S' \neq S$ with probability ε_S at the end of each period. In state S, let q_S be the quantity a seller produces for the asset, and let $V_{1,S}$ and $V_{0,S}$ be the value functions. For $S = A$,

$$\begin{aligned} V_{1,A} &= \alpha_1 [u(q_A) + \beta(1-\varepsilon_A) V_{0,A} + \beta\varepsilon_A V_{0,B}] \\ &\quad + (1-\alpha_1)\beta[(1-\varepsilon_A)V_{1,A} + \varepsilon_A V_{1,B}] + \rho \end{aligned} \tag{14}$$

$$\begin{aligned} V_{0,A} &= \alpha_0 [\beta(1-\varepsilon_A) V_{1,A} + \beta\varepsilon_A V_{1,B} - c(q_A)] \\ &\quad + (1-\alpha_0)\beta\left[(1-\varepsilon_A)V_{0,A} + \varepsilon_A V_0^B\right], \end{aligned} \tag{15}$$

and similarly for $S = B$. In words, (14) says a buyer may or may not trade for q, but in either case the state changes with some probability, and he always gets ρ.

It is always possible that agents ignore S, but a proper sunspot equilibrium has $q_A \neq q_B$, say $q_B > q_A$. Emulating the analysis in the baseline model, we let $\Delta_S = V_{1,S} - V_{0,S}$ and determine the terms of trade by $v(q_S) = \beta(1-\varepsilon_S)\Delta_S + \beta\varepsilon_S\Delta_{S'}$. Using (14)-(15), this implies

$$v(q_A) = \varepsilon_A F(q_B) + (1-\varepsilon_A) F(q_A) \text{ and } v(q_B) = \varepsilon_B F(q_A) + (1-\varepsilon_B) F(q_B),$$

where $F(q) \equiv \beta\alpha_1 u(q) + \beta\alpha_0 c(q) + \beta\rho + \beta(1-\alpha_1-\alpha_0) v(q)$. Following the original sunspot work of Azariadis (1981), we solve these for

$$\varepsilon_A = \frac{v(q_A) - F(q_A)}{F(q_B) - F(q_A)} \text{ and } \varepsilon_A = \frac{F(q_B) - v(q_B)}{F(q_B) - F(q_A)}. \tag{16}$$

If we can find $(q_A, q_B, \varepsilon_A, \varepsilon_B)$ such that $q_B > q_A$ and $\varepsilon_A, \varepsilon_B \in (0,1)$, where the ε's are given by (16), we have satisfied all the conditions for a proper sunspot equilibrium.

It is convenient here to work with (v_A, v_B) rather than (q_A, q_B), with $v_S \equiv v(q_S)$, and rewrite (16) as

$$\varepsilon_A = \frac{v_A - \Psi(v_A)}{\Psi(v_B) - \Psi(v_A)} \text{ and } \varepsilon_A = \frac{\Psi(v_B) - v_B}{\Psi(v_B) - \Psi(v_A)}, \tag{17}$$

where $\Psi(v) = F \circ v^{-1}(v)$. Notice $\Psi(v) = \beta\Phi(v/\beta)$, where Φ is defined in Section 3, so $\Psi(v)$ is concave (convex) iff $\Phi(\Delta)$ is concave (convex); indeed, $v_t = \Psi(v_{t+1})$ is simply the dynamical system in v space rather than q space, and a solution to $v = \Psi(v)$ is a SME.

Proposition 8 *Let $v > 0$ be a stable SME. Then $\forall (v_A, v_B)$ in a neighborhood around v with $v_A < v < v_B$ there is a proper sunspot with the $\varepsilon_A, \varepsilon_B \in (0,1)$ given by (17).*

Proof: We seek (v_A, v_B) such that $\varepsilon_A, \varepsilon_B \in (0,1)$ when the ε's are given by (17). In the the left panel of Figure 6, there are two SME, v_L and $v_H > 0$, and the lower one is stable. In this case, pick any $v_A \in (\underline{v}, \bar{v})$, where $\underline{v}$ is the maximum of 0, another SME to the left of v_L if it exists, and the horizontal intercept of Ψ. Now pick any $v_B \in (v_L, \bar{v})$ where $\bar{v}$ is the minimum of the next SME to the

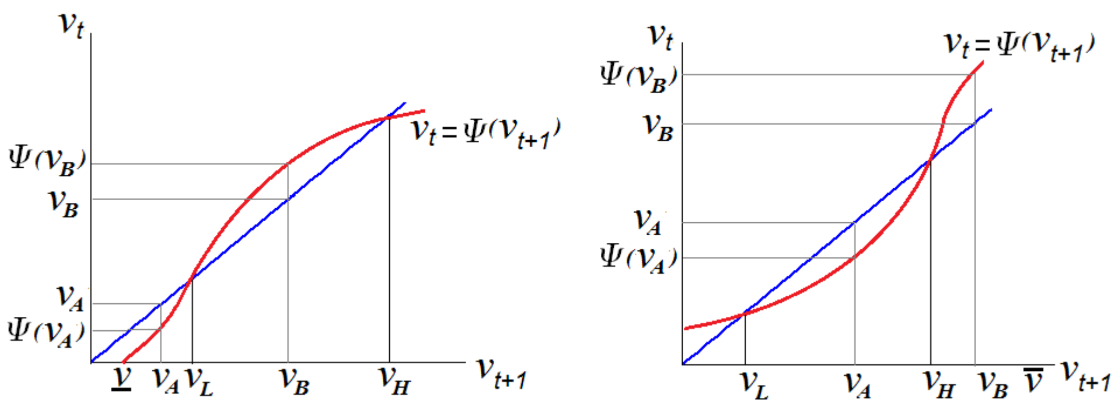

Figure 6: Existence of ISE

right of v_L or the v corresponding to the upper bound $\bar{q}$. From the graph, clearly, $\Psi(v_B) > v_B > v_A > \Psi(v_A)$, which is easily shown to imply that the ε's given in (17) are in $(0, 1)$. The right panel shows the case where the higher SME v_H is stable, which is similar. ■

5.2 Lotteries

As is well known, in economies with nonconvexities, including indivisible assets, it may be desirable to trade using lotteries – e.g., see Hansen (1984) or Rogerson (1988) in labor, and Berentsen et al. (2002,2004) or Berentsen and Rocheteau (2007) in monetary models. Building on that work, without loss in generality we restrict attention to the case where sellers deliver the goods with probability 1 while buyers hand over their assets with probability $\pi \leq 1$, and it is possible to have $q < q^*$ and $\pi = 1$, or $q = q^*$ and $\pi < 1$, but not $q < q^*$ and $\pi < 1$ or $q > q^*$.[14]

With lotteries, for any π_t we have

$$V_{1t} = \alpha_1 \left[u(q_t) - \beta \pi_t \Delta_{t+1} \right] + \beta V_{1t+1} + \rho \tag{18}$$

$$V_{0t} = \alpha_0 \left[\pi_t \beta \Delta_{t+1} - c(q_t) \right] + \beta V_{0t+1}, \tag{19}$$

Consider generalized Nash bargaining (other bargaining solutions and posting are

[14]To be clear, the claim is not that $q > q^*$ is impossible in all related models (see Berentsen et al. 2004), but it is impossible in those under consideration here.

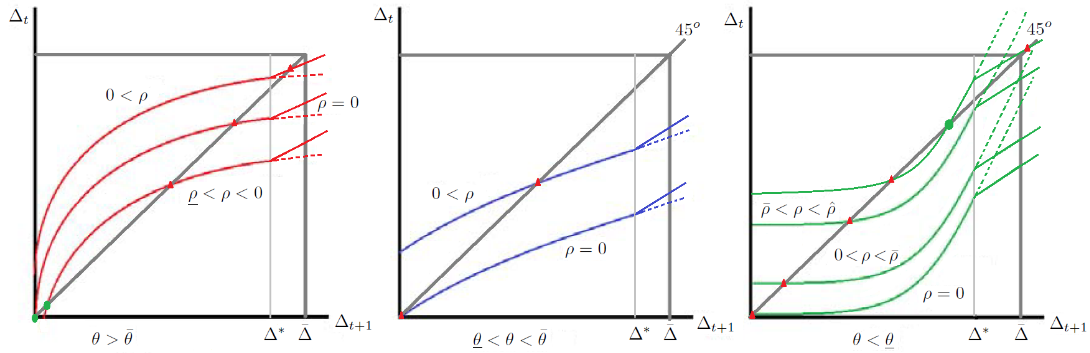

Figure 7: DME with lotteries

similar),

$$\max_{q_t,\pi_t} \left[u\left(q_t\right)-\pi_t\beta\Delta_{t+1}\right]^{\theta}\left[\pi_t\beta\Delta_{t+1}-c\left(q_t\right)\right]^{1-\theta} \text{ st } \pi_t \leq 1,$$

given the constraints $q_t, \pi_t \geq 0$ are slack, as they must be if there are gains from trade. When $\beta\Delta_{t+1} < \theta c\left(q^*\right)+(1-\theta)\,u\left(q^*\right)$ the solution is $\pi = 1$ and q is the same as without lotteries; otherwise it is $q = q^*$ and

$$\pi_t = \frac{\theta c\left(q^*\right)+(1-\theta)\,u\left(q^*\right)}{\beta\Delta_{t+1}}. \tag{20}$$

A difference from the benchmark model is that now there is an equilibrium with trade at $q > 0$ no matter how big ρ gets: rather than hoarding the asset, when ρ is big, buyers use it to acquire $q = q^*$ by offering it to the seller with probability $\pi < 1$, with $\pi \to 0$ as $\rho \to \infty$. So asset circulation slows down as ρ rises but never stops.

For a general mechanism $v\left(q_t\right) = \pi_t\beta\Delta_{t+1}$, with lotteries the dynamical system is $\Delta_t = \Phi^*\left(\Delta_{t+1}\right)$, where

$$\begin{aligned}\Phi^*\left(\Delta\right) \equiv\ & \alpha_1 u \circ Q^*\left(\beta\Delta\right)+\alpha_0 c \circ Q^*\left(\beta\Delta\right)+\rho \\ & +\left[1-\left(\alpha_1+\alpha_0\right)\pi^*\left(\beta\Delta\right)\right]\beta\Delta\end{aligned} \tag{21}$$

with $Q^*\left(\beta\Delta\right) = Q\left(\beta\Delta\right)$ and $\pi^*\left(\beta\Delta\right) = 1$ if $Q\left(\beta\Delta\right) < q^*$, while $Q^*\left(\beta\Delta\right) = q^*$ and $\pi\left(\beta\Delta\right) = v\left(q^*\right)/\beta\Delta^*$ otherwise. Thus Φ^* is linear for $\Delta_{t+1} > \Delta^* = v\left(q^*\right)/\beta$. Figure 7 amends Figure 2 to allow lotteries (dashed curves reproducing the case without lotteries). When there is two SME without lotteries, either the higher one is eliminated (as in the second highest curve in the right panel), or it survives and we introduce another SME with $\Delta > \Delta^*$ (as in the highest curve in the right panel). But the main point is that lotteries do not eliminate multiplicity: the examples at the end of Section 4 have $\Delta^s < \Delta^*$, and they do not change when lotteries are allowed.

6 Conclusion

We analyzed stationary and dynamic equilibria with indivisible assets, allowing a general meeting technology, arbitrary return ρ, and either generic bargaining or posting. These components appear individually in previous work, but there is no prior treatment of the general case.[15] Dynamics with posting is especially novel, and relevant because it has solid microfoundations. But we can also use our results to rationalize earlier work. Trejos and Wright (1995), e.g., study dynamics with Nash bargaining, which is problematic if we want to interpret this as the limit of strategic bargaining, but not if we reinterpret it as competitive search.

In terms of technical results, a few proofs are nontrivial, e.g. Lemma 4, and some are directly applicable in other contexts, e.g. verification of the SOC's for competitive search. In terms of substantive results, all the findings enhance our understanding of a workhorse monetary model, but one that we find particularly interesting is the nonexistence of cyclic or chaotic equilibria. Given such equilibria exist in some models, including search models with divisible money, we conjectured they would also exist here – indeed, we thought constructing them would be easier with indivisible money. We were wrong. Deterministic fluctuations cannot occur here, even if there are other sorts of exotic dynamics, like sunspot equilibria. This is something monetary theorists ought to know.

[15] Trejos and Wright (2016), e.g., study dynamics with bargaining, but only Kalai bargaining, and a special meeting technology, while Julien et al. (2008) study posting, but only with a special meeting technology, and only consider steady state.

Appendices

A. Solving problem (10): The Lagrangian is

$$\mathcal{L} = \alpha\left(b_t\right)\left[\beta\Delta_{t+1} - c\left(q_t\right)\right] + \beta V_{1t+1} + \lambda_t \left\{ \frac{\alpha\left(b_t\right)}{b_t}\left[u(q_t) - \beta\Delta_{t+1}\right] + \beta V_{1t+1} - V_{1t} + \rho \right\}$$

where $\lambda_t > 0$ is the multiplier. The FOC's are:

$$b_t \quad : \quad \alpha'\left(b_t\right)\left[\beta\Delta_{t+1} - c\left(q_t\right)\right] + \frac{\lambda_t\left[u(q_t) - \beta\Delta_{t+1}\right]\left[b_t\alpha'\left(b_t\right) - \alpha\left(b_t\right)\right]}{b_t^2} = 0 \tag{22}$$

$$q_t \quad : \quad -\alpha\left(b_t\right)c'\left(q_t\right) + \frac{\lambda_t\alpha\left(b_t\right)u'\left(q_t\right)}{b_t} = 0 \tag{23}$$

$$\lambda_t \quad : \quad \frac{\alpha\left(b_t\right)}{b_t}\left[u(q_t) - \beta\Delta_{t+1}\right] + \beta V_{1t+1} - V_{1t} + \rho = 0 \tag{24}$$

From (23), $\lambda_t = b_t c'\left(q_t\right)/u'\left(q_t\right)$. Then from (22),

$$\varepsilon\left(b_t\right)\left[\beta\Delta_{t+1} - c\left(q_t\right)\right]u'\left(q_t\right) = \left[1 - \varepsilon\left(b_t\right)\right]\left[u(q_t) - \beta\Delta_{t+1}\right]c'\left(q_t\right),$$

where $\varepsilon\left(b\right) = b\alpha'\left(b\right)/\alpha\left(b\right)$. This can be rearranged into (12).

Next, after simplification, the bordered Hessian at any solution to the FOC's is given by

$$H = \begin{bmatrix} \frac{\alpha''(u-\beta\Delta)}{\varepsilon}\frac{c'}{u'} + \frac{2\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2}\frac{c'}{u'} & -\frac{\alpha c'}{b} & -\frac{\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2} \\ -\frac{\alpha c'}{b} & \frac{\alpha(c'u''-u'c'')}{u'} & -\frac{\alpha u'}{b} \\ -\frac{\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2} & -\frac{\alpha u'}{b} & 0 \end{bmatrix}.$$

The determinant is

$$|H| = -\left(\frac{\alpha}{b}\right)^2\left(u - \beta\Delta\right)\left[\frac{\alpha''c'u'}{\varepsilon} + \frac{\alpha\left(1-\varepsilon\right)^2\left(u-\beta\Delta\right)\left(c'u''-u'c''\right)}{b^2u'}\right] > 0.$$

Hence the SOC's hold at any solution to the FOC's. This implies there is a unique solution to the FOC's. ■

B. Proof of Lemma 4: First notice that R has a critical point at any q where the denominator $D = (r+\alpha_0)u\left(q\right) - \alpha_0 c\left(q\right) - \rho$ vanishes. This happens when $z\left(q\right) = \rho$ where $z\left(q\right) \equiv (r+\alpha_0)u\left(q\right) - \alpha_0 c\left(q\right)$. As shown in Figure 8, $z\left(0\right) = 0$, $z\left(\bar{q}\right) = \bar{\rho}$ and $z''\left(q\right) < 0$. The left panel shows the case $\bar{q} > \hat{q} \equiv \arg\max z\left(q\right)$; the right panel shows $\bar{q} < \hat{q}$. In the left panel the following is clear: $\rho < 0$ implies no solution to $z\left(q\right) = \rho$ and hence no critical points in $[0, \bar{q}]$; $\rho = 0$ implies one critical point at $q = 0$; $0 < \rho < \bar{\rho}$ implies one critical point $\tilde{q}_1 \in (0, \hat{q})$; $\bar{\rho} < \rho < \hat{\rho}$ implies two critical points $\tilde{q}_1 \in (0, \hat{q})$ and $\tilde{q}_2 \in (\hat{q}, \bar{q})$; and $\rho > \hat{\rho}$ implies no critical points. The right panel is similar except $\tilde{q}_2 > \bar{q}$, which simply means the case $\rho > \bar{\rho}$ is irrelevant. Also note that when $\rho \in (\bar{\rho}, \hat{\rho})$ and $R(q)$ has two critical points in $(0, \bar{q})$, its numerator satisfies $N > 0 \; \forall q \in (0, \bar{q})$, while $D > 0$ and hence $R(q) > 0$ iff $q \in (\tilde{q}_1, \tilde{q}_2)$.

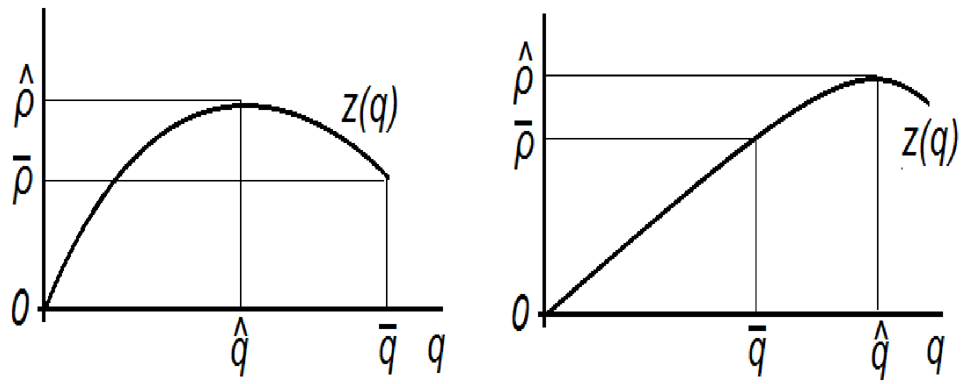

Figure 8: The function $z(q)$

With these preliminary results in hand, we now go through an argument for each panel of Figure 4. To begin, derive

$$R'(q) = \frac{(r + \alpha_1 + \alpha_0) \left\{ \rho \left[c'(q) - u'(q) \right] + r \left[c(q) u'(q) - u(q) c'(q) \right] \right\}}{\left[(r + \alpha_0) u(q) - \alpha_0 c(q) - \rho \right]^2}. \tag{25}$$

Note that $c(q) u'(q) < u(q) c'(q)$ for any concave u and convex c with $u(0) = c(0) = 0$. Therefore $R'(q) < 0$ when $\rho = 0$, as shown in the upper left panel of Figure 4.

Next note that

$$\begin{aligned}
\lim_{q \to 0} R'(q) &= \lim_{q \to 0} \frac{(r + \alpha_1 + \alpha_0) \left\{ \rho \left[\frac{c'(q)}{u'(q)} - 1 \right] + r \left[c(q) - \frac{u(q) c'(q)}{u'(q)} \right] \right\}}{\left[(r + \alpha_0) \frac{u(q)}{u'(q)} - \alpha_0 \frac{c(q)}{u'(q)} - \frac{\rho}{u'(q)} \right]^2 u'(q)} \\
&= \lim_{q \to 0} \frac{-(r + \alpha_1 + \alpha_0)}{\rho} u'(q).
\end{aligned}$$

Hence, $\rho > 0 \Rightarrow R'(0) = -\infty$ and $\rho < 0 \Rightarrow R'(0) = +\infty$. Also,

$$R'(\bar{q}) = \frac{(r + \alpha_1 + \alpha_0) \left[c'(\bar{q}) - u'(\bar{q}) \right]}{(\rho - \bar{\rho})}, \tag{26}$$

so $\rho > \bar{\rho} \Rightarrow R'(\bar{q}) > 0$ and $\rho < \bar{\rho} \Rightarrow R'(\bar{q}) < 0$. Now, as in the lower left panel of Figure 4, with $0 < \rho < \bar{\rho}$, we claim $R'(q) < 0 \; \forall q \in (\tilde{q}_1, \bar{q})$ such that $R(q) > 0$.

From (25) this is obvious for $q < q^*$ since then $u'(q) > c'(q)$. It remains to show $R'(q) < 0$ for $q > \max\{\tilde{q}_1, q^*\}$ such that $R(q) > 0$. In this range, D and N are positive, and N is concave with a maximum at $q < q^*$.

Consider the right panel of Figure 8 with $\hat{q} > \bar{q}$. For $q \in (q^*, \bar{q})$, it is clear that D is increasing and N is decreasing, so $R'(q) < 0$ when $R(q) > 0$, because $D, N < 0$. Now consider the left panel with $\hat{q} < \bar{q}$. Suppose $\hat{q} < q^*$. Then it is again clear that D is increasing and N is decreasing, so $R'(q) < 0$ for $q \in (q^*, \hat{q})$ when $R(q) > 0$. Now consider $(\hat{q}, \bar{q})$. For that, notice

$$\begin{aligned} R'(q) &\propto \rho \left[c'(q) - u'(q) \right] + r \left[c(q) u'(q) - u(q) c'(q) \right] \\ &< \left[c'(q) - u'(q) \right] \left[\rho - ru(q) \right] < 0, \end{aligned} \tag{27}$$

where the last inequality follows because $ru(q) > ru(\bar{q}) = \bar{\rho}$ for $q \in (\hat{q}, \bar{q})$. Finally, consider $\hat{q} < q^*$. Again, we only need to show $R'(q) < 0$ for $q > q^*$, which is true by (27) for $\bar{q} > q > \max(\hat{q}, q^*)$. This completes the argument for lower left panel of Figure 4.

Continuing with the upper and lower right panels, we note that $R''(q)$ is messy, in general, but at any q such that $R'(q) = 0$ it is easy to check

$$R'' = \frac{\left[(rc - \rho) u'' - (ru - \rho) c'' \right] (r + \alpha_1 + \alpha_0)}{D^2}.$$

In the upper right panel with $\rho < 0$ and $D > 0$, this implies $R''(q) < 0$ when $R'(q) = 0$, so R increases below and decreases above a unique point $\tilde{q} \in (0, \bar{q})$. Similarly, in the lower right panel with $\rho > \bar{\rho}$, we know $R'' > 0$ when $R'(q) = 0$ over the relevant range. Hence, R decreases below and decreases above a unique $\tilde{q} \in (\tilde{q}_1, \tilde{q}_2)$. This completes the argument for the upper and lower right panels, and thus completes the proof. ■

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