A Note on Complicated Dynamics in Simple Models of Liquidity*

Chao He

East China Normal University

Randall Wright

University of Wisconsin - Madison, FRB Chicago and FRB Minneapolis

January 25, 2018

Abstract

We analyze stationary and nonstationary equilibria in search-based models of liquidity with indivisible assets. Two formulations are considered: the usual one in monetary theory, based on random search and bargaining; and a more novel one, with better microfoundations, based on directed search and price posting. For each we generalize earlier specifications and prove new results. These models have equilibria where endogenous variables change over time as self-fulfilling prophecies, including sunspot equilibria, as previously shown in special cases. As not previously shown, and as may be surprising, we prove in general there are no equilibria where variables cycle deterministically.

Key words: dynamics, liquidity, search, cycles

JEL classification numbers: D83, E50, E44

*We thank Ken Burdett, Han Han, Alberto Trejos, Yiyuan (Edward) Xie and Yu Zhu for input. Wright acknowledges support from the Ray Zemon Chair in Liquid Assets at the Wisconsin School of Business. The usual disclaimers apply.

1 Introduction

We analyze dynamics in simple – i.e., indivisible-asset – search models of liquidity. Two formulations are studied: the usual one in monetary theory, based on random search and bargaining as in Shi (1995) or Trejos and Wright (1995); and a more novel one, with arguably better microfoundations, based on directed search and price posting as in Julien et al. (2008). For each we generalize assumptions in previous analyses and prove new results. The models can have multiple steady states, plus equilibria where endogenous variables change over time as self-fulfilling prophecies, including stochastic (sunspot) equilibria that fluctuate randomly. This is well known (for bargaining; not for posting). A reasonable conjecture is that the models also have cyclic or chaotic equilibria where they fluctuate deterministically. We prove in a fairly general specification that this conjecture is false, which may be surprising, given what is known about monetary economics in general.[1]

We are interested in indivisible-asset models that restrict individual holdings to $m \in \{0, 1\}$ because they allow one to succinctly address key issues: What frictions make monetary exchange an equilibrium or an efficient arrangement? When are assets valued for liquidity? Do liquidity considerations lead to multiplicity or volatility? How does this impinge on allocations and welfare? While some of these issues can be studied when $m \in \{0, 1, ...\}$ or $m \in \mathbb{R}_+$, that is complicated by the endogenous distribution of m across agents, and hence requires special assumptions or numerical methods (see Lagos et al. 2017 for survey of the literature). Hence, we think there is still a role for indivisible-asset theory.[2]

[1] See Rocheteau and Wright (2013) for cycles in a modern monetary model, and Azariadis (1993) for similar results in OLG (overlapping generatons) and other older models. In a setting more similar to ours, Burdett et al. (2018) can get cycles, for reasons explained below.

[2] To put this in perspective, indivisible-asset models are deployed to good effect in studies of middlemen by Rubinstein and Wolinsky (1987), banking by Cavalcanti and Wallace (1999), and OTC financial markets by Duffie et al. (2005). Similarly, the Pissarides (2000) and Burdett and Coles (1997) models of employment and partnership formation assume a firm can hire n workers and an agent can have n partners, where $n \in \{0, 1\}$, yet they still provide important insights. Indeed, $\{0, 1\}$ restrictions have proved useful in search theory going back at least to Diamond (1982). Also note that $\{0, 1\}$ is not a bad approximation for many assets, e.g., housing, but most of the papers make this assumption not for realism, only to illustrate salient ideas transparently.

Given this, we want to better understand the properties of these models. Again, we relax several assumptions in previous analyses and provide more results. Our characterization of existence, multiplicity vs uniqueness, and dynamics extends previous papers by using general bargaining solutions and meeting technologies, plus we consider assets with general rates of return, not just currency. While Trejos and Wright (2016) allow general returns, they only consider a special parametric meeting technology, and only examine Kalai bargaining, while our specification nests Kalai, Nash, simple strategic bargaining, Walrasian pricing, and other solution concepts. Similarly, while Julien et al. (2008) study price posting with directed search, they also consider a special parametric meeting technology, only allow fiat money, and only examine steady state. So there is much here that is new.

As regards the bigger picture, one may ask, why is it interesting to know if a given model can or cannot have cyclical monetary equilibria? Well, it is a venerable notion that monetary economies are subject to multiplicity, instability or volatility, in ways that nonmonetary economies are not, and economists have spent some time investigating this (again see Lagos et al. 2017). This is parallel to the banking literature, where it is sometimes said that economies with financial intermediation face similar issues. In work emanating from Diamond and Dybvig (1983), first someone finds assumptions under which bank runs occur, then someone shows with slightly different assumptions they cannot, someone else shows with a twist on the assumptions they can after all, etc. (Ennis and Keister 2010 nicely summarize this evolution). Similarly, the results here refine our knowledge about when monetary exchange is or is not susceptible to volatility: in a widely-used model, while multiple equilibria are possible, we prove cycles are not. We also explain the differences between this model and ones where cycles are possible.

In what follows, Section 2 describes the environment. Then Sections 3 and 4 analyze bargaining and posting. Section 5 concludes. More technical results are in the Appendix; extensions and alternative specifications are contained in a Supplementary Appendix posted at https://hechao.weebly.com/.

2 Environment

A $[0,1]$ continuum of infinitely-lived agents meet bilaterally in discrete time. There is a set of goods $\mathcal{G}$, where agents of type i consume only a subset $\mathcal{G}^i$ and produce $g \notin \mathcal{G}^i$. The measure of any type is the same. From q units of $g \in \mathcal{G}^i$, i gets $u(q)$ utils while production costs $c(q)$ utils, with $u(0) = c(0) = 0$ and $u(\bar{q}) = c(\bar{q}) > 0$ for $\bar{q} > 0$. Also, $u'(q), c'(q) > 0$, $u''(q) < 0$, and $c''(q) \geq 0$ $\forall q > 0$, $u'(0) = \infty$. Also, $q^* \in (0, \bar{q})$ solves $u'(q^*) = c'(q^*)$. Then specialization is generally captured as follows: drawing at random two agents i and j, δ is the probability i produces $g \in \mathcal{G}^j$ and j produces $g^j \in \mathcal{G}^i$ (a double coincidence), while σ is the probability i produces $g \in \mathcal{G}^j$ and j produces $g^j \notin \mathcal{G}^i$ (a single coincidence). For simplicity, set $\delta = 0$ to rule out barter, and let goods in $\mathcal{G}$ be nonstorable so they cannot act as commodity money. Then, to preclude credit, assume agents cannot commit and histories are private information.

The above formulation is standard in models of asset liquidity. Here there is one asset, with a flow return ρ, in utils. If $\rho > 0$ it is a dividend, as in standard asset-pricing theory; if $\rho < 0$ it is a cost of holding the asset, as in many models of commodity money; and if $\rho = 0$ it is fiat currency according to common usage. While much work has focused on $\rho = 0$, clearly, it is interesting to relax that in theory and in applications. In the class of environments under consideration, an individual's asset position is $m \in \{0, 1\}$. With a fixed supply $M \in (0, 1)$, at any point in time measure M of agents, called buyers, have $m = 1$, while measure $1 - M$, called sellers, have $m = 0$. After trade a buyer becomes a seller and vice versa.

For the meeting process, as in Pissarides (2000), consider any market with a measure n_1 of buyers and a measure n_0 of sellers, where $b = n_1/n_0$ denotes market tightness. The number of buyer-seller meetings is $\mu = \mu(n_1, n_0) \leq \min(n_1, n_0)$, so a seller's probability of meeting a buyer is $\alpha_0 = \mu(n_1, n_0)/n_0$ and a buyer's probability of meeting a seller is $\alpha_1 = \mu(n_1, n_0)/n_1$. As usual, we assume $\mu(\cdot)$ displays constant returns (one can use increasing returns but, as remarked in Section 5, that misses the whole point). Hence, $\alpha_0 = \mu(b, 1) \equiv \alpha(b)$ and $\alpha_1 = \alpha(b)/b$, where $\alpha(0) = 0$, $\alpha'(b) > 0$ and $\alpha''(b) < 0$. This extends the typical specification in monetary theory, $\mu = An_1n_0/(n_1 + n_0)$.[3] This is too restrictive for our purposes, as it implies $\alpha_0 + \alpha_1 \leq 1$, while as explained below it is possible and interesting to relax that – e.g., frictionless matching with $M = 1/2$ implies $\alpha_0 + \alpha_1 = 2$.

In fact, what matters for i is the probability of meeting a seller that produces $g \in \mathcal{G}^i$. Usually that is $\alpha_0\sigma$ for sellers and $\alpha_1\sigma$ for buyers, where σ is the single-coincidence probability. But to better compare environments suppose, as in Matsui and Shimizu (2005), agents can always direct their search toward the right type, although whether they meet one is still randomly determined by $\mu(\cdot)$. Then for the difference across specifications below we can focus on bargaining vs posting.

3 Bargaining

Let the value functions for buyers and sellers be V_{1t} and V_{0t}, and let $\Delta_t \equiv V_{1t} - V_{0t}$. Then the standard discrete-time Bellman equations are

$$V_{1t} = \alpha_1 [u(q_t) + \beta V_{0t+1}] + (1 - \alpha_1) \beta V_{1t+1} + \rho \tag{1}$$

$$V_{0t} = \alpha_0 [\beta V_{1t+1} - c(q_t)] + (1 - \alpha_0) \beta V_{0t+1}, \tag{2}$$

where $\beta = 1/(1 + r)$, $r > 0$.[4] Here q_t is determined by a *generic bargaining solution* $\beta\Delta_{t+1} = v(q_t)$, where $v(0) = 0$, $v'(q) > 0$, and $c(q) \leq v(q) \leq u(q)$ $\forall q \in [0, \bar{q}]$. Thus, q is a function of the gain from acquiring the asset, $\beta\Delta$. This nests many solution concepts including the standard generalization of Nash (1950),

$$\max_{q_t} [u(q_t) - \beta\Delta_{t+1}]^{\theta} [-c(q_t) + \beta\Delta_{t+1}]^{1-\theta},$$

where θ is buyers' bargaining power. From the FOC we get $\beta\Delta_{t+1} = v(q_t)$ where

$$v(q_t) = \frac{\theta u'(q_t) c(q_t) + (1 - \theta) c'(q_t) u(q_t)}{\theta u'(q_t) + (1 - \theta) c'(q_t)}. \tag{3}$$

[3] An exception is Julien et al. (2008), who use $\mu = An_0 [1 - \exp(-b)]$, which is nice but also quite special. See also Coles (1999).

[4] Intuitively, (1) says this: with probability α_1 a buyer meets a seller, consumes q_t and continues without the asset; with probability $1 - \alpha_1$ he does not trade and continues with the asset; and in any event he enjoys ρ. The story for (2) is similar.

Similarly, Kalai's (1977) bargaining solution is $\beta\Delta_{t+1} = v(q_t)$ where[5]

$$v(q_t) = \theta c(q_t) + (1-\theta) u(q_t). \tag{4}$$

To determine Δ_t, subtract (1)-(2) to get

$$\Delta_t = \alpha_1 u(q_t) + \alpha_0 c(q_t) + \rho + (1 - \alpha_1 - \alpha_0) \beta\Delta_{t+1}. \tag{5}$$

This implies $q_t = Q(\beta\Delta_{t+1})$, where $Q'(\beta\Delta_{t+1}) = 1/v'(q_t) > 0$. Then (5) defines a difference equation $\Delta_t = \Phi(\Delta_{t+1})$, where

$$\Phi(\Delta_{t+1}) \equiv \alpha_1 u \circ Q(\beta\Delta_{t+1}) + \alpha_0 c \circ Q(\beta\Delta_{t+1}) + \rho + (1 - \alpha_1 - \alpha_0) \beta\Delta_{t+1}, \tag{6}$$

and $f \circ g(x) \equiv f[g(x)]$ denotes the composite. In steady state (5) implies

$$\beta\Delta = \frac{\alpha_1 u(q) + \alpha_0 c(q) + \rho}{r + \alpha_1 + \alpha_0}, \tag{7}$$

where the RHS is a weighted average of buyers' utility $u(q)$, sellers' cost $c(q)$, and the payoff to holding the asset forever, ρ/r. For stationary outcomes, combine (7) with $\beta\Delta = v(q)$ to get $\rho = e(q)$, where

$$e(q) \equiv (r + \alpha_1 + \alpha_0) v(q) - \alpha_1 u(q) - \alpha_0 c(q). \tag{8}$$

A stationary monetary equilibrium, or SME, is defined by $\beta\Delta^s = v(q^s)$, where $q^s \in (0, \bar{q}]$ solves $\rho = e(q^s)$. Equivalently, it is described by $\Delta^s \in (0, \bar{\Delta}]$, where $\bar{\Delta}$ solves $\beta\bar{\Delta} = v(\bar{q})$. A dynamic monetary equilibrium, or DME, is a nonconstant solution to (6) with $\Delta_t \in (0, \bar{\Delta}]\ \forall t$, or to (5) with $q_t \in (0, \bar{q}]\ \forall t$, after eliminating $\beta\Delta_{t+1} = v(q_t)$. We call these *monetary* equilibria, even if the asset is not fiat currency unless $\rho = 0$, because the asset *circulates* as a medium of exchange.[6]

[5] Note (4) is not the definition of Kalai bargaining, but a formula that follows from his axioms, just like (3) follows from Nash's axoims. Nash and Kalai differ away from $q = q^*$, in general, and the latter has several advantages in models with liquidity considerations (Aruoba et al. 2007). Other examples are given in Gu and Wright (2016), who also derive properties of $v(q)$ axiomatically, and Zhu (2016), who shows how it emerges from simple stragetic bargaining.

[6] In SME we need $q^s \in (0, \bar{q}]$ for $c(q) \leq u(q)$, and hence for voluntary trade. It may be less obvious that we need $q_t \in (0, \bar{q}]\ \forall t$ in DME, but as shown below, q_t paths that exit $(0, \bar{q}]$ never return, so we may as well define DME this way (at least for nonstochastic equilibria, because with sunspots q_t can exit and return; see Trejos and Wright 2016 and our Supplementary Appendix). Also, while we focus on monetary outcomes, $\rho \leq 0$ implies there is an equilibrium where agents dispose of assets, and $\rho > 0$ with ρ big implies there is one where they hoard rather than trade assets (at least if we rule out lotteries; see Berentsen et al. 2013 and our Supplementary Appendix).

Previous analyses of SME impose particular bargaining solutions – e.g., Shi (1995) or Trejos and Wright (1995) use symmetric Nash, or a game that amounts to the same thing; Rupert et al. (2001) use generalized Nash; and Trejos and Wright (2016) use Kalai. To facilitate the presentation, let us start with Kalai. Then these properties of $e(q)$ follow from direct calculation:

Lemma 1 *Assume Kalai bargaining, and define* $\underline{\theta} \in (0,1)$ *and* $\bar{\theta} \in (\underline{\theta},1)$ *by*

$$\bar{\theta} \equiv \frac{r+\alpha_0}{r+\alpha_1+\alpha_0} \text{ and } \underline{\theta} \equiv \frac{\alpha_0}{r+\alpha_1+\alpha_0}. \tag{9}$$

Then (1) $\theta > \bar{\theta} \Rightarrow e'' > 0$*; (2)* $\underline{\theta} < \theta < \bar{\theta} \Rightarrow e' > 0$*; and (3)* $\theta < \underline{\theta} \Rightarrow e'' < 0$*. Also,* $e(0) = 0 < e(\bar{q}) = ru(\bar{q})$*.*

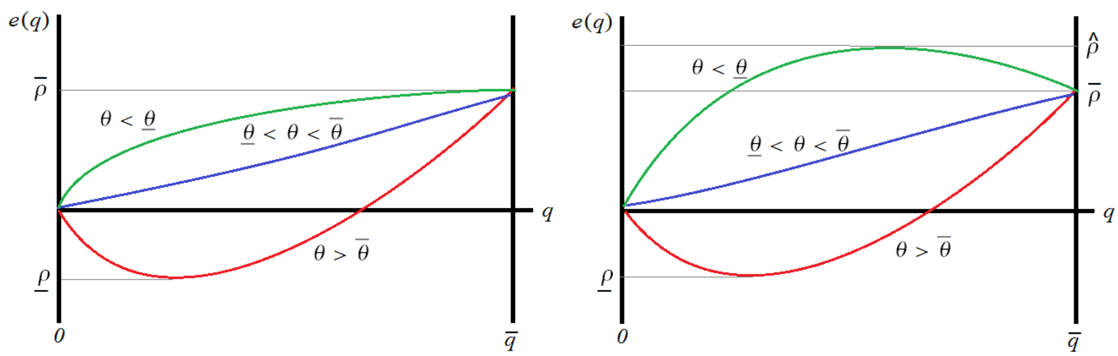

Figure 1: SME with Kalai bargaining

Figure 1 shows $e(q)$ for the three cases in Lemma 1 using this notation: $\bar{\rho} \equiv e(\bar{q})$; $\hat{\rho} \equiv \max_{[0,\bar{q}]} e(q)$ when $\theta < \underline{\theta}$; and $\underline{\rho} \equiv \min_{[0,\bar{q}]} e(q) < 0$ when $\theta > \bar{\theta}$. The left panel shows $\hat{\rho} = \bar{\rho}$, the right shows $\hat{\rho} > \bar{\rho}$. From this, the following is immediate:

Proposition 1 *Assume Kalai bargaining. (1)* $\theta > \bar{\theta}$*. If* $\rho < \underline{\rho}$ *there is no SME; if* $\underline{\rho} < \rho < 0$ *there are two SME; if* $0 \leq \rho < \bar{\rho}$ *there is one SME; if* $\rho > \bar{\rho}$ *there is no SME. (2)* $\underline{\theta} < \theta < \bar{\theta}$*. If* $\rho \leq 0$ *there is no SME; if* $0 < \rho < \bar{\rho}$ *there is one SME; if* $\rho > \bar{\rho}$ *there is no SME. (3)* $\theta < \underline{\theta}$*. If* $\rho \leq 0$ *there is no SME; if* $0 < \rho \leq \bar{\rho}$ *there is one SME; if* $\rho > \bar{\rho}$ *there are two subcases: (3a) if* $\hat{\rho} = \bar{\rho}$ *(left panel of Figure 1)*

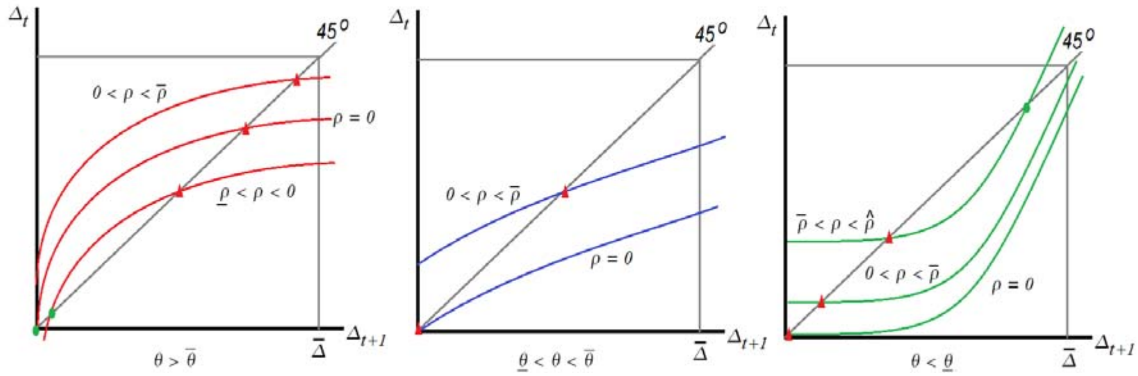

Figure 2: DME with Kalai bargaining

there is no SME, and (3b) if $\hat{\rho} > \bar{\rho}$ *(the right panel) there are two SME if* $\bar{\rho} < \rho < \hat{\rho}$ *and no SME if* $\rho > \hat{\rho}$.

As regards stability, the next result is obvious from Figure 2, showing (6) in (Δ_{t+1}, Δ_t) space for different θ and ρ, with stable steady states indicated by dots and unstable ones by triangles:

Lemma 2 *Assume Kalai bargaining. (1)* $\theta > \bar{\theta}$. *If* $\underline{\rho} < \rho < 0$ *the high SME is unstable and the low one is stable; if* $\rho = 0$ *the unique SME is unstable and* $(0, 0)$ *is stable; and if* $0 < \rho < \bar{\rho}$ *the unique SME is unstable. (2)* $\underline{\theta} < \theta < \bar{\theta}$. *The unique SME is unstable. (3)* $\theta < \underline{\theta}$. *If* $0 < \rho \leq \bar{\rho}$ *the unique SME is unstable; and if* $\bar{\rho} < \rho < \hat{\rho}$ *the low SME is unstable while the high one is stable.*

Based on this, the following characterization of DME is immediate:

Proposition 2 *Assume Kalai bargaining. If there are no SME then there are no DME. Otherwise: (1)* $\theta > \bar{\theta}$. *If* $\underline{\rho} < \rho < 0$ *there are DME with* Δ_t *approaching the low SME for any initial* Δ_0 *in an interval; if* $\rho = 0$ *there are DME with* Δ_t *approaching* 0 *for any* Δ_0 *in an interval; and if* $0 < \rho < \bar{\rho}$ *there are no DME. (2)* $\underline{\theta} < \theta < \bar{\theta}$. *There are no DME. (3)* $\theta < \underline{\theta}$. *If* $\bar{\rho} < \rho < \hat{\rho}$ *there are DME with* Δ_t *approaching the high SME for any* Δ_0 *in an interval.*

In terms of economics, in DME, Δ_t and q_t vary over time as self-fulfilling prophecies. This cannot happen in the middle panel of Figure 2, where there is a unique SME, it is unstable, and any path other than steady state eventually leaves $[0, \bar{\Delta}]$ never to return. It can happen when there is a stable SME, in which case paths leading to it from any initial Δ_0 are DME. The basin of attraction for SME can be large: in the left panel of Figure 2, e.g., given $\underline{\rho} < \rho < 0$, for any Δ_0 below the high SME there is a DME converging to the low SME. In terms of mathematics, except for the fact that Trejos and Wright (2016) use continuous time, Proposition 2 extends their results to general $\mu(\cdot)$. While that is hardly a major contribution, it does allow $\Phi'(\Delta^s) < 0$, while the usual $\mu(\cdot)$ does not, and that makes more challenging the results on cyclical equilibria discussed below. Also, $\Phi'(\Delta^s) < 0$ is needed for some interesting outcomes.[7] So generalizing $\mu(\cdot)$ does have merit.

At this point it is not hard to go from Kalai to generic bargaining. Consider first $\rho \geq 0$. We still have $e(0) = 0 < e(\bar{q})$ for any $v(q)$. This implies existence of SME when $\rho \in (0, \hat{\rho})$ and nonexistence when $\rho > \hat{\rho}$. However, for $\rho \in (0, \bar{\rho})$ we cannot say that there is exactly one SME, only a generically odd number, and for $\rho \in (\bar{\rho}, \hat{\rho})$ we cannot say that there are exactly two SME, only a generically even number. This is no surprise – of course a parametric $v(\cdot)$ provides more structure and hence more precise results. In any case, consider next $\rho \leq 0$, and notice

$$e'(q) = (r + \alpha_1 + \alpha_0) v'(q) - \alpha_1 u'(q) - \alpha_0 c'(q),$$

Now $e'(0) < 0$ implies existence of SME $\forall \rho \in (\underline{\rho}, 0)$ and nonexistence for $\rho < \underline{\rho}$.[8] Given the set of SME, it is easy to describe DME graphically, as in Figure 2.

Proposition 3 *Except for the exact number of SME, the results for Kalai generalize to any $v(q)$.*

[7]Here are a few: (i) there can be DME starting from Δ_0 above the high SME; (ii) given Δ_0 there can be multiple DME; and (iii) there can be DME that oscillate around the high SME before converging to the low SME. And these are not just hypothetical issues – with a general $\mu(\cdot)$ it is easy to construct examples with $\Phi'(\Delta^s) < 0$, even if it is impossible for the usual $\mu(\cdot)$.

[8]For $e'(0) < 0$ we need a condition on $v'(0)$, obviously, such as $(r + \alpha_1 + \alpha_0) v'(0) < \alpha_1 u'(0)$. This does rule out some bargaining solutions – e.g., there is no SME with $\rho \leq 0$ and take-it-or-leave-it offers by sellers, since then $v(q) = u(q)$.

Now it is natural to conjecture there exist DME that are limit cycles around SME – but that is false for any bargaining solution representable as $v(q_t)$.

Proposition 4 *Cyclic equilibria do not exist.*

Proof: First, $u', c', Q' > 0$ and $\alpha_0, \alpha_1 \in (0, 1]$ imply

$$\begin{aligned} \Phi'(\Delta_{t+1}) &= \alpha_1 \beta u' Q' + \alpha_0 \beta c' Q' + (1 - \alpha_1 - \alpha_0) \beta \\ &> (1 - \alpha_1 - \alpha_0) \beta > -1. \end{aligned}$$

Given this and $\Phi'(0) > 0$, which follows from (6), it is clear that Φ and Φ^{-1} cannot cross off the 45^o line. So there cannot be cycles of period 2. But the Sharkovskii Theorem says that cycles of period $n > 2$ imply cycles of period 2. Hence there are no cycles of any period. ■

We remark that the Sharkovskii Theorem is a standard tool used to show when cycles exist (e.g., Azariadis 1993); here it is used to show they do not exist. Also, if $\Phi'(\Delta) > 0$ then the dynamics are monotone and cyclic equilibria obviously do not exit. That is the case for the usual $\mu(\cdot)$, where $\alpha_0 = AM$ and $\alpha_1 = A(1 - M)$. For a general matching technology $\Phi'(\Delta) > 0$ is not true, but $\Phi'(\Delta) > -1$ is, and that is sufficient.

4 Posting

Posting with directed search, also called *competitive search*, has sellers committing to terms of to attract buyers. Thus, in addition to consumers of good g knowing where to find the right producers, for any g the market further segments into submarkets identified by pairs (q, b), where agents commit to trade q for the asset if they meet, and tightness b determines the meeting probabilities. As before, $\alpha_0 = \alpha(b)$ and $\alpha_1 = \alpha(b)/b$, except b is now tightness in a given submarket, and while equilibrium implies $b = M/(1 - M)$ in every active submarket, to find equilibrium we first allow b to vary across submarkets (see the survey by Wright et al. 2016 for details).

Competitive search has arguably better microfoundations than bargaining, especially for dynamics. To explain, first, of course one can always impose a cooperative bargaining solution like Nash or Kalai, but one might worry about its strategic foundations. Consider a standard game where a buyer and seller in a stationary setting make counteroffers of q until one is accepted, with the time between offers denoted $\eta > 0$ (Rubinstein 1982). The unique subgame-perfect equilibrium has the first offer accepted, and labeling it q^η to indicate its dependence on timing, $q^\eta \to q^N$ as $\eta \to 0$ where q^N is the generalized Nash solution (Binmore et al. 1986). Similarly, Dutta (2012) provides strategic foundations for Kalai. Such results are commonly regarded as providing support for cooperative bargaining – indeed the quest for such strategic foundations has been dubbed the Nash program.[9]

Coles and Wright (1998), Ennis (2001) and Coles and Muthoo (2003) show the result in Binmore et al. (1986) holds in dynamic models if we focus on steady state but *not* otherwise. When $\eta \to 0$, in fact, the limit of subgame-perfect equilibrium is a path for q satisfying a differential equation with q^N as a stationary point, but generally $q \neq q^N$ on nonstationary paths.[10] Those papers also argue that using Nash out of steady state is tantamount to using an extensive-form game where agents are myopic, negotiating as if conditions were constant, when they are not. And it matters for results: with forward-looking strategic bargaining, e.g., Coles and Wright (1998) construct dynamic equilibria that are not possible with Nash. However, their construction is complex. An advantage of posting is that it is simple even with strategic forward-looking agents, making it ideal for our application.

To proceed, notice that still the value functions satisfy (1)-(2), Δ_t satisfies (5) and steady state satisfies (7). But now, as is standard, to determine (q, b) we

[9] Serrano (2005) nicely puts is as follows: "Similar to the microfoundations of macroeconomics, which aim to bring closer the two branches of economic theory, the Nash program is an attempt to bridge the gap between the two counterparts of game theory (cooperative and non-cooperative). This is accomplished by investigating non-cooperative procedures that yield cooperative solutions as their equilibrium outcomes."

[10] One can show $q = q^N$ out of steady state in some special cases, e.g., $\theta = 1$, $\theta = 0$ or $u(q) = c(q) = q$, but these are far too restrictive for our purposes.

maximize sellers' expected payoff subject to buyers' getting their expected market payoff, which is an equilibrium object but taken as given by individuals:

$$V_{0t} = \max_{(q_t,b_t)} \left\{ \alpha\left(b_t\right)\left[\beta\Delta_{t+1} - c\left(q_t\right)\right] + \beta V_{0t+1} \right\} \tag{10}$$

$$\text{st } \frac{\alpha\left(b_t\right)}{b_t}\left[u(q_t) - \beta\Delta_{t+1}\right] + \rho + \beta V_{1t+1} = V_{1t} \tag{11}$$

The Appendix shows that the SOC's hold at any interior solution to the FOC's, implying a unique such solution. So all active submarkets are the same, or equivalently, given $\mu(\cdot)$ displays CRS, there is just one submarket for each good.

Moreover, the FOC's imply

$$\beta\Delta_{t+1} = \frac{\varepsilon\left(b_t\right)u'\left(q_t\right)c\left(q_t\right) + \left[1-\varepsilon\left(b_t\right)\right]c'\left(q_t\right)u\left(q_t\right)}{\varepsilon\left(b_t\right)u'\left(q_t\right) + \left[1-\varepsilon\left(b_t\right)\right]c'\left(q_t\right)} \tag{12}$$

where $\varepsilon \equiv b\alpha'(b)/\alpha(b)$ is the elasticity, and in equilibrium $b = M/(1-M)$. Note that, consistent with other applications of competitive search, (12) is identical to what one gets with Nash bargaining if we swap ε for bargaining power θ, which is convenient, since any results we get for competitive search also hold for Nash.

As in Section 3, write (12) as $\beta\Delta_{t+1} = v\left(q_t\right)$, invert to get $q_t = Q\left(\beta\Delta_{t+1}\right)$, and substitute it into (5) to get $\Delta_t = \Phi\left(\Delta_{t+1}\right)$, just like (6). However, the method here is different, because the $v\left(q_t\right)$ implied by (12) is more complicated. To proceed, equate (7) to (12) and rearrange to get

$$\frac{(1-\varepsilon)\,c'(q)}{\varepsilon u'(q)} = \frac{\alpha_1 u(q) - (r+\alpha_1)\,c(q) + \rho}{(r+\alpha_0)u(q) - \alpha_0 c(q) - \rho}. \tag{13}$$

Call the LHS $L(q)$ and the RHS $R(q)$, so SME solves $L(q) = R(q)$. In general, their properties are complicated, but Lemma 3 in the Appendix shows the situation must look like one of the panels in Figure 3. Given that, the following result is immediate:

Proposition 5 *Assume posting. For $\rho \in [0, \bar{\rho})$, SME exists uniquely. For $\rho < 0$, multiple SME exist if $|\rho|$ is not too big and no SME exist if $|\rho|$ is too big. For $\rho > \bar{\rho}$, multiple SME exist if ρ is not too big and no SME exist if ρ is too big.*

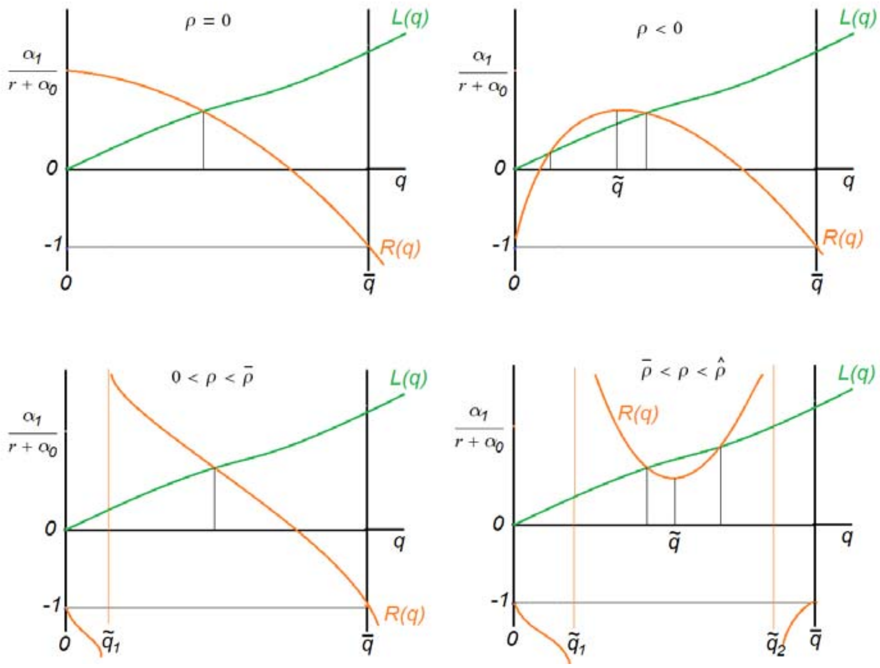

Figure 3: Four possible cases for the function $R(q)$

Proof: A SME is a solution to $R(q) = L(q)$ with $q \in (0, \bar{q})$. In the top left panel of Figure 3, with $\rho = 0$, SME exists because $R(0) > L(0)$ and $R(\bar{q}) < L(\bar{q})$, and it is unique because $R'(q) < 0$. In the bottom left panel, with $\rho \in (0, \bar{\rho})$, there is a critical point $\tilde{q}_1 \in (0, \bar{q})$, and SME exists at $q \in (\tilde{q}_1, \bar{q})$ because $R(q) > L(q)$ for q near $\tilde{q}_1$ and $R(\bar{q}) < L(\bar{q})$. It is unique because $R'(q) < 0$ when $R(q) > 0$. In the bottom right, with $\rho \in (\bar{\rho}, \hat{\rho})$ there are two critical points $\tilde{q}_1$ and $\tilde{q}_2$, and we have multiple SME when ρ is not too big. However, when ρ is too big, $R(q)$ shifts up too much and there is no SME. Finally, in the top right panel, with $\rho < 0$, there are multiple SME when $|\rho|$ is not too big. However, if $|\rho|$ is too big $R(q)$ shifts down too much and there is no SME. ■

For dynamics, Figure 4 immediately yields an analog to Proposition 2:

Proposition 6 *Assume posting. If there are no SME then there are no DME. Otherwise: (a) For $\rho = 0$ there are DME with Δ_t approaching 0 from any Δ_0 in an interval. (b) For $\rho < 0$ there are DME with Δ_t approaching every stable SME from*

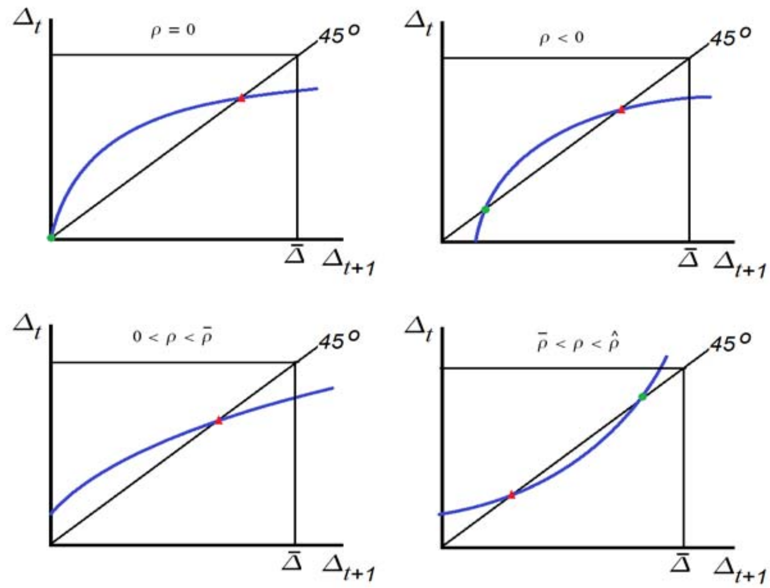

Figure 4: DME with directed search and posting

any Δ_0 in an interval. (c) For $\rho \in (0, \bar{\rho})$ there are no DME. (d) For $\rho \in (\bar{\rho}, \hat{\rho})$ there are DME with Δ_t approaching every SME for any Δ_0 in an interval.

Next, we exploit the formal equivalence between competitive search and generalized Nash to significantly extend previous results.

Proposition 7 *Assume generalized Nash bargaining. Then SME and DME are exactly as described in Propositions 5 and 6.*

Finally, we know there are no cycles with posting by virtue of Proposition 4, since it applies to any bargaining solution $v(q)$, including generalized Nash, which is formally equivalent to posting.

Proposition 8 *Deterministic cyclic equilibria do not exist with posting.*

In a sense, this "salvages" the discussion of dynamics in Trejos and Wright (1995) with Nash bargaining, which we argued is problematic if one wants to interpret it as the limit of strategic bargaining. It is not problematic if we reinterpret it in terms of competitive search, and that delivers the same equations.

5 Conclusion

It is interesting to know if monetary models have cyclical equilibria because there is a view that economies featuring monetary exchange are particularly prone to volatility.[11] In a standard, logically-consistent environment that has an endogenous role for liquidity – i.e., assets are used as media of exchange – we show that multiple stationary and dynamic equilibria are possible but cycles are not. This was not obvious ex ante, since cycles can arise in other monetary models, including OLG models (recall fn. 1). In those settings it is routine to derive the analog to our dynamical system, say $z_t = \Phi(z_{t-1})$ where z is real balances. In the textbook OLG model, we can get $\Phi'(z^s) < -1$ and hence cyclic equilibria if (at the risk of oversimplifying) savings or labor supply functions are "backward bending" (see Azariadis 1993 for a textbook treatment). We do not have those ingredients, but we have others those models lack, the meeting technology $\mu(\cdot)$ and the trading protocol $v(\cdot)$, and we made every effort to specify these quite generally. Can those ingredients substitute for "backward bending" savings or labor supply functions?

We prove they cannot. At least not without "tricks" like increasing returns in $\mu(\cdot)$, which are well known to generate cycles (e.g., Diamond and Fudenberg 1989), but that has nothing to do with money and the issue at hand is whether monetary considerations per se lead to volatility. This raises a question: how do we get stochastic – i.e., sunspot – equilibria? These are shown to exist in the Supplementary Appendix using a well-known method (see Trejos and Wright 2016 for a recent application), although not the one used in the textbook OLG model. In an OLG model, $\Phi'(z^s) < -1$ implies the existence of cyclic and stochastic equilibria around a steady state where $\Phi(z^s)$ cuts the 45^o from above. We cannot appeal to that because $\Phi'(q^s) < -1$ is impossible. Instead, we construct equilibria that

[11] To be clear, the goal is not to build models capturing deterministic cycles that we think we see in the data; it is to show that some economies can be unstable in the sense that they can generate intricate dynamics due to beliefs, not (only) to changes in fundamentals. Rudimentary models do not generate serious predictions testable by time-series econometrics. Yet if they can display intricate dynamics based on beliefs, it suggests that realistically-complex economies can, too.

stochastically fluctuate around z^s when $\Phi(q^s)$ cuts the 45^o line from below. This occurs in the low (high) steady state in the far left (right) panel of Figure 2 in the bargaining model. Figure 4 shows something similar in the posting model. In these scenarios, we get stochastic but not cyclic equilibria.

Finally, Burdett et at. (2018) propose a model that is similar to ours in many ways, including $m \in \{0, 1\}$, but also has a key difference: it uses noisy search as in Burdett and Judd (1983). This means that a buyer each period sees one or more than one posted price with positive probability. That can yield cycles. Indeed, as is very well known, Burdett-Judd search models generate many interesting phenomena, including endogenous price dispersion. This is relevant because $\Phi'(q^s) < -1$ is possible in Burdett et at. (2018), not because savings or labor supply functions are "backward bending" as in OLG models, but because of nonlinear feedback from q_{t+1} to the endogenous price distribution at t and hence q_t. That channel is inoperative here, where the price distribution is degenerate, so $\Phi'(q^s) < -1$ and hence cycles are impossible even though sunspots are possible. We realize that we are going into a lot of detail here, but this is how we discover which features of models generate different kinds of results, and in particular different kinds of volatility. For us, that constitutes progress.

References

[1] S. Aruoba, G. Rocheteau and C. Waller (2007) "Bargaining and the Value of Money," *JME* 54, 2636-55.

[2] C. Azariadis (1993) *Intertemporal Macroeconomics*.

[3] A. Berentsen, M. Molico and R. Wright (2002) "Indivisibilities, Lotteries and Monetary Exchange," *JET* 107, 70-94.

[4] K. Burdett and M. Coles (1997) "Marriage and Class," *QJE* 112, 141-68.

[5] K. Burdett and K. Judd (1983) "Equilibrium Price Dispersion," *Econometrica* 51, 955-70.

[6] K. Burdett, A. Trejos and R. Wright (2018) "A New Suggestion for Simplifying the Theory of Money," *JET*, in press.

[7] K. Binmore, A. Rubinstein and A. Wolinsky (1986) "The Nash Bargaining Solution in Economic Modelling," *Rand Journal* 17, 176-88.

[8] M. Coles (1999) "Turnover Externalities with Marketplace Trading," *IER* 40, 851-68.

[9] M. Coles and A. Muthoo (2003) "Bargaining in a Non-stationary Environment," *JET* 109, 70-89.

[10] M. Coles and R. Wright (1998) "A Dynamic Model of Search, Bargaining, and Money," *JET* 78, 32-54.

[11] D. Diamond and P. Dybvig (1983) "Bank Runs, Deposit Insurance, and Liquidity," *JPE* 91, 401-19.

[12] P. Diamond (1982) "Aggregate Demand Management in Search Equilibrium," *JPE* 90, 881-94.

[13] P. Diamond and D. Fudenberg (1989) "Rational Expectations Business Cycles in Search Equilibrium," *JPE* 97, 606-619.

[14] D. Duffie, N. Gârleanu and L. Pederson (2005) "Over-the-Counter Markets," *Econometrica* 73, 1815-1847.

[15] R. Dutta (2012) "Bargaining with Revoking Costs," *GEB* 74, 144-153.

[16] H. Ennis (2001) "On Random Matching, Monetary Equilibria, and Sunspots," *Macro Dynamics* 5, 132-142.

[17] H. Ennis and T. Keister (2010) "On the Fundamental Reasons for Bank Fragility," FRB Richmond Economic Quarterly 96, 33-58.

[18] C. Gu and R. Wright (2016) "Monetary Mechanisms," *JET*, 163, 644-657.

[19] B. Julien, J. Kennes and I. King (2008) "Bidding For Labor," *RED* 3, 619-649.

[20] E. Kalai (1977) "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," *Econometrica* 45, 1623-30.

[21] R. Lagos, G. Rocheteau and R. Wright (2017) "Liquidity: A New Monetarist Perspective," *JEL* 55, 371-440.

[22] A. Matsui and T. Shimizu (2005) "A Theory of Money with Market Places," *IER* 46, 35-59.

[23] J. Nash (1950) "The Bargaining Problem," *Econometrica* 18, 155-162.

[24] C. Pissarides (2000) *Equilibrium Unemployment Theory*. Cambridge.

[25] G. Rocheteau and R. Wright (2013) "Liquidity and Asset Market Dynamics," *JME* 60, 275-94.

[26] P. Rupert, M. Schindler and R. Wright (2001) "Generalized Search-Theoretic Models of Monetary Exchange," *JME* 48, 605-22.

[27] A. Rubinstein (1982) "Perfect Equilibrium in a Bargaining Model," *Econometrica* 50, 97-109.

[28] A. Rubinstein and A. Wolinsky (1987) "Middlemen," *QJE* 102, 581-94.

[29] R. Serrano (2005) "Fifty Years of the Nash Program, 1953-2003," *Investigaciones Economicas* 29, 219-258.

[30] S. Shi (1995) "Money and Prices: A Model of Search and Bargaining," *JET* 67, 467-496.

[31] A. Trejos and R. Wright (1995) "Search, Bargaining, Money, and Prices," *JPE* 103, 118-141.

[32] A. Trejos and R. Wright (2016) "Search-Based Models of Money and Finance: An Integrated Approach," *JET* 164, 10-31 .

[33] R. Wright, B. Julien, P. Kircher and V. Guerrieri (2016) "Directed Search: A Guided Tour," mimeo.

[34] Y. Zhu (2016) "Strategic Bargaining in Models of Money and Credit," mimeo.

Appendix

A. Here we solve the problem (10). Form the Lagrangian

$$\mathcal{L} = \alpha(b_t)[\beta\Delta_{t+1} - c(q_t)] + \beta V_{1t+1} + \lambda_t \left\{ \frac{\alpha(b_t)}{b_t}[u(q_t) - \beta\Delta_{t+1}] + \beta V_{1t+1} - V_{1t} + \rho \right\}$$

where $\lambda_t > 0$ is the multiplier. The FOC's are:

$$b_t \; : \; \alpha'(b_t)[\beta\Delta_{t+1} - c(q_t)] + \frac{\lambda_t[u(q_t) - \beta\Delta_{t+1}][b_t\alpha'(b_t) - \alpha(b_t)]}{b_t^2} = 0 \quad (14)$$

$$q_t \; : \; -\alpha(b_t)c'(q_t) + \frac{\lambda_t\alpha(b_t)u'(q_t)}{b_t} = 0 \quad (15)$$

$$\lambda_t \; : \; \frac{\alpha(b_t)}{b_t}[u(q_t) - \beta\Delta_{t+1}] + \beta V_{1t+1} - V_{1t} + \rho = 0 \quad (16)$$

From (15), $\lambda_t = b_t c'(q_t)/u'(q_t)$. Then from (14),

$$\varepsilon(b_t)[\beta\Delta_{t+1} - c(q_t)]u'(q_t) = [1 - \varepsilon(b_t)][u(q_t) - \beta\Delta_{t+1}]c'(q_t),$$

where $\varepsilon(b) = b\alpha'(b)/\alpha(b)$. This can be rearranged into (12).

After simplification the bordered Hessian at any solution to the FOC's is

$$H = \begin{bmatrix} \frac{\alpha''(u-\beta\Delta)}{\varepsilon}\frac{c'}{u'} + \frac{2\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2}\frac{c'}{u'} & -\frac{\alpha c'}{b} & -\frac{\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2} \\ -\frac{\alpha c'}{b} & \frac{\alpha(c'u''-u'c'')}{u'} & -\frac{\alpha u'}{b} \\ -\frac{\alpha(1-\varepsilon)(u-\beta\Delta)}{b^2} & -\frac{\alpha u'}{b} & 0 \end{bmatrix}.$$

The determinant is

$$|H| = -\left(\frac{\alpha}{b}\right)^2 (u - \beta\Delta)\left[\frac{\alpha''c'u'}{\varepsilon} + \frac{\alpha(1-\varepsilon)^2(u-\beta\Delta)(c'u''-u'c'')}{b^2u'}\right] > 0.$$

Hence the SOC's hold at any solution to the FOC's, so there is a unique solution to the FOC's. ■

B. We verify $L(q)$ and $R(q)$ are as shown in Figure 3. First, clearly $L(0) = 0$ and $L'(q) > 0$ $\forall q \in (0, \bar{q})$. Next, let us dispense with $\rho = \bar{\rho}$, in which case $q = \bar{q}$ is a SME. Now consider $\rho \neq \bar{\rho}$. It is easy to see $R(\bar{q}) = -1$, $R(0) = -1$ if $\rho \neq 0$ and $R(0) = \alpha_1/(r + \alpha_0)$ if $\rho = 0$. For the rest, we have this:

Lemma 3 *(a) As shown in the upper left panel of Figure 3, $\rho = 0$ implies $R'(q) < 0$ $\forall q$. (b) As shown in the upper right panel, $\rho < 0$ implies there exists $\tilde{q} \in (0, \bar{q})$ such that $R'(q) > 0$ $\forall q < \tilde{q}$ and $R'(q) < 0$ $\forall q > \tilde{q}$. (c) As shown in the lower left,*

$0 < \rho < \bar{\rho}$ implies there exists $\tilde{q}_1 \in (0, \bar{q})$ such that $R(q) < 0 \; \forall q < \tilde{q}_1$. Also $R(q) > 0 \Rightarrow R'(q) < 0 \; \forall q > \tilde{q}_1$. Also, $R(q) \nearrow \infty$ as $q \searrow \tilde{q}_1$. (d) As shown in the lower right, $\bar{\rho} < \rho < \hat{\rho}$ implies there exist $\tilde{q} \in (0, \bar{q})$, $\tilde{q}_1 \in (0, \tilde{q})$ and $\tilde{q}_2 \in (\tilde{q}, \bar{q})$ such that $R(q) > 0 \; \forall q \in (\tilde{q}_1, \tilde{q}_2)$, $R(q) < 0 \; \forall q \notin (\tilde{q}_1, \tilde{q}_2)$, $R'(q) < 0 \; \forall q < \tilde{q}$ and $R'(q) > 0 \; \forall q > \tilde{q}$. Also, $R(q) \nearrow \infty$ as $q \searrow \tilde{q}_1$ or $q \nearrow \tilde{q}_2$.

Proof: First note that R has a critical point at any q where the denominator $D = (r + \alpha_0)u(q) - \alpha_0 c(q) - \rho$ vanishes. This happens when $z(q) = \rho$ where $z(q) \equiv (r + \alpha_0)u(q) - \alpha_0 c(q)$. As shown in Figure 5, $z(0) = 0$, $z(\bar{q}) = \bar{\rho}$ and $z''(q) < 0$. The left panel shows the case $\bar{q} > \hat{q} \equiv \arg\max z(q)$; the right panel shows $\bar{q} < \hat{q}$. In the left panel the following is clear: $\rho < 0$ implies no solution to $z(q) = \rho$ and hence no critical points in $[0, \bar{q}]$; $\rho = 0$ implies one critical point at $q = 0$; $0 < \rho < \bar{\rho}$ implies one critical point $\tilde{q}_1 \in (0, \hat{q})$; $\bar{\rho} < \rho < \hat{\rho}$ implies two critical points $\tilde{q}_1 \in (0, \hat{q})$ and $\tilde{q}_2 \in (\hat{q}, \bar{q})$; and $\rho > \hat{\rho}$ implies no critical points. The right panel is similar except $\tilde{q}_2 > \bar{q}$, which simply means the case $\rho > \bar{\rho}$ is irrelevant. Also note that when $\rho \in (\bar{\rho}, \hat{\rho})$ and $R(q)$ has two critical points in $(0, \bar{q})$, its numerator satisfies $N > 0 \; \forall q \in (0, \bar{q})$, while $D > 0$ and hence $R(q) > 0$ iff $q \in (\tilde{q}_1, \tilde{q}_2)$.

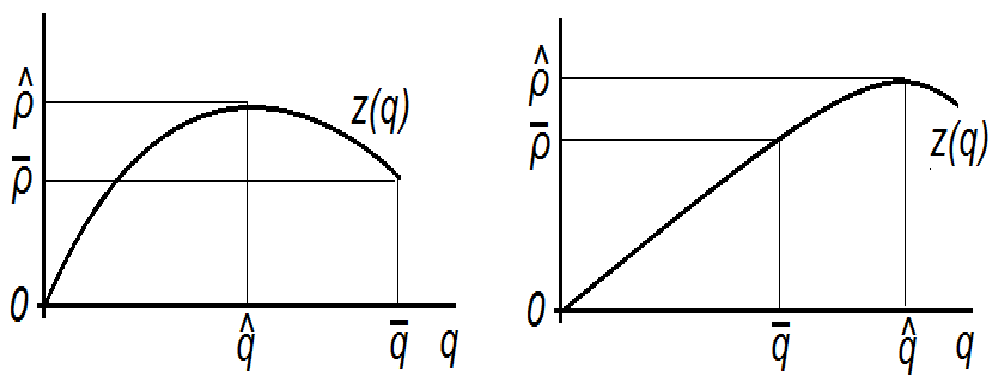

Figure 5: The function $z(q)$

We now go through an argument for each panel of Figure 3. To begin, derive

$$R'(q) = \frac{(r + \alpha_1 + \alpha_0)\{\rho[c'(q) - u'(q)] + r[c(q)u'(q) - u(q)c'(q)]\}}{[(r + \alpha_0)u(q) - \alpha_0 c(q) - \rho]^2}. \tag{17}$$

Note that $c(q)u'(q) < u(q)c'(q)$ for any concave u and convex c with $u(0) = c(0) = 0$. So $R'(q) < 0$ when $\rho = 0$, as in the upper left panel of Figure 3.

Next note that

$$\lim_{q\to 0} R'(q) = \lim_{q\to 0} \frac{(r+\alpha_1+\alpha_0)\left\{\rho\left[\frac{c'(q)}{u'(q)}-1\right]+r\left[c(q)-\frac{u(q)c'(q)}{u'(q)}\right]\right\}}{\left[(r+\alpha_0)\frac{u(q)}{u'(q)}-\alpha_0\frac{c(q)}{u'(q)}-\frac{\rho}{u'(q)}\right]^2 u'(q)}$$
$$= \lim_{q\to 0} \frac{-(r+\alpha_1+\alpha_0)}{\rho} u'(q).$$

Hence, $\rho > 0 \Rightarrow R'(0) = -\infty$ and $\rho < 0 \Rightarrow R'(0) = +\infty$. Also,

$$R'(\bar{q}) = \frac{(r+\alpha_1+\alpha_0)\left[c'(\bar{q}) - u'(\bar{q})\right]}{(\rho - \bar{\rho})}, \tag{18}$$

so $\rho > \bar{\rho} \Rightarrow R'(\bar{q}) > 0$ and $\rho < \bar{\rho} \Rightarrow R'(\bar{q}) < 0$. Now, as in the lower left panel, with $0 < \rho < \bar{\rho}$, we claim $R'(q) < 0$ $\forall q \in (\tilde{q}_1, \bar{q})$ such that $R(q) > 0$. From (17) this is obvious for $q < q^*$ since then $u'(q) > c'(q)$. It remains to show $R'(q) < 0$ for $q > \max\{\tilde{q}_1, q^*\}$ such that $R(q) > 0$. In this range, D and N are positive, and N is concave with a maximum at $q < q^*$.

Consider the right panel of Figure 5 with $\hat{q} > \bar{q}$. For $q \in (q^*, \bar{q})$, it is clear that D is increasing and N is decreasing, so $R'(q) < 0$ when $R(q) > 0$, because $D, N < 0$. Now consider the left panel with $\hat{q} < \bar{q}$. Suppose $\hat{q} < q^*$. Then it is again clear that D is increasing and N is decreasing, so $R'(q) < 0$ for $q \in (q^*, \hat{q})$ when $R(q) > 0$. Now consider $(\hat{q}, \bar{q})$. For that, notice

$$R'(q) \propto \rho\left[c'(q) - u'(q)\right] + r\left[c(q)u'(q) - u(q)c'(q)\right]$$
$$< \left[c'(q) - u'(q)\right]\left[\rho - ru(q)\right] < 0, \tag{19}$$

where the last inequality follows because $ru(q) > ru(\bar{q}) = \bar{\rho}$ for $q \in (\hat{q}, \bar{q})$. Finally, consider $\hat{q} < q^*$. Again, we only need to show $R'(q) < 0$ for $q > q^*$, which is true by (19) for $\bar{q} > q > \max(\hat{q}, q^*)$. This completes the argument for lower left panel.

Continuing with the upper and lower right panels, we note that $R''(q)$ is messy, in general, but at any q such that $R'(q) = 0$ it is easy to check

$$R'' = \frac{\left[(rc-\rho)u'' - (ru-\rho)c''\right](r+\alpha_1+\alpha_0)}{D^2}.$$

In the upper right panel with $\rho < 0$ and $D > 0$, this implies $R''(q) < 0$ when $R'(q) = 0$, so R increases below and decreases above a unique point $\tilde{q} \in (0, \bar{q})$. Similarly, in the lower right panel with $\rho > \bar{\rho}$, we know $R'' > 0$ when $R'(q) = 0$ over the relevant range. Hence, R decreases below and decreases above a unique $\tilde{q} \in (\tilde{q}_1, \tilde{q}_2)$. This completes the argument for the upper and lower right panels. ■

Supplementary (not for publication) Appendix

A. Although deterministic cycles are impossible, there can be sunspot equilibria, where Δ_t and q_t fluctuate randomly. This has been shown in related indivisible-asset models by several people, although there are differences (e.g., continuous vs discrete time). Here we show how to derive these kind of results, using our notation for a general meeting technology $\mu(\cdot)$ and trading protocol $v(\cdot)$.

Consider a random variable $S \in \{A, B\}$, where S switches from S to $S' \neq S$ with probability ε_S at the end of each period. In state S, q_S is the quantity a seller produces for the asset, while $V_{1,S}$ and $V_{0,S}$ are the value functions. For $S = A$,

$$\begin{aligned} V_{1,A} &= \alpha_1 \left[u(q_A) + \beta(1-\varepsilon_A) V_{0,A} + \beta\varepsilon_A V_{0,B} \right] \qquad (20) \\ &\quad + (1-\alpha_1)\beta\left[(1-\varepsilon_A)V_{1,A} + \varepsilon_A V_{1,B}\right] + \rho \\ V_{0,A} &= \alpha_0 \left[\beta(1-\varepsilon_A) V_{1,A} + \beta\varepsilon_A V_{1,B} - c(q_A) \right] \qquad (21) \\ &\quad + (1-\alpha_0)\beta\left[(1-\varepsilon_A)V_{0,A} + \varepsilon_A V_{0,B}\right], \end{aligned}$$

and similarly for $S = B$. Thus, a buyer may or may not trade for q, but in either case the state changes with some probability, and in any event he gets ρ.

It is possible that agents ignore S, but a proper sunspot equilibrium has $q_A \neq q_B$, say $q_B > q_A$. Emulating the analysis in the baseline model, we let $\Delta_S = V_{1,S} - V_{0,S}$ and determine the terms of trade by $v(q_S) = \beta(1-\varepsilon_S)\Delta_S + \beta\varepsilon_S\Delta_{S'}$. Using (20)-(21), this implies

$$v(q_A) = \varepsilon_A F(q_B) + (1-\varepsilon_A) F(q_A) \text{ and } v(q_B) = \varepsilon_B F(q_A) + (1-\varepsilon_B) F(q_B),$$

where $F(q) \equiv \beta\alpha_1 u(q) + \beta\alpha_0 c(q) + \beta\rho + \beta(1-\alpha_1-\alpha_0) v(q)$. Following Azariadis (1981), we solve these for

$$\varepsilon_A = \frac{v(q_A) - F(q_A)}{F(q_B) - F(q_A)} \text{ and } \varepsilon_A = \frac{F(q_B) - v(q_B)}{F(q_B) - F(q_A)}. \qquad (22)$$

If we can find $(q_A, q_B, \varepsilon_A, \varepsilon_B)$ such that $q_B > q_A$ and $\varepsilon_A, \varepsilon_B \in (0, 1)$, where the ε's are given by (22), we have satisfied the conditions for a proper sunspot equilibrium.

It is convenient here to work with (v_A, v_B) rather than (q_A, q_B), with $v_S \equiv v(q_S)$, and rewrite (22) as

$$\varepsilon_A = \frac{v_A - \Psi(v_A)}{\Psi(v_B) - \Psi(v_A)} \text{ and } \varepsilon_A = \frac{\Psi(v_B) - v_B}{\Psi(v_B) - \Psi(v_A)}, \qquad (23)$$

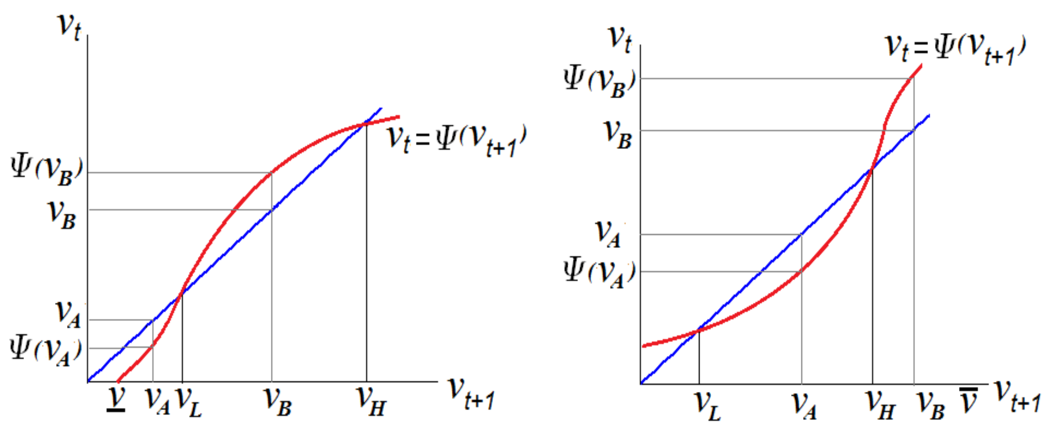

Figure 6: Existence of ISE

where $\Psi(v) = F \circ v^{-1}(v)$. Notice $\Psi(v) = \beta\Phi(v/\beta)$, where Φ is defined in Section 3, so $\Psi(v)$ is concave (convex) iff $\Phi(\Delta)$ is concave (convex); indeed, $v_t = \Psi(v_{t+1})$ is simply the dynamical system in v space rather than q space, and a solution to $v = \Psi(v)$ is a SME.

Proposition 9 *Let $v > 0$ be a stable SME. Then $\forall (v_A, v_B)$ in a neighborhood around v with $v_A < v < v_B$ there is a proper sunspot with the $\varepsilon_A, \varepsilon_B \in (0,1)$ given by (23).*

Proof: We seek (v_A, v_B) such that $\varepsilon_A, \varepsilon_B \in (0,1)$ when the ε's are given by (23). In the the left panel of Figure 6, there are two SME, v_L and $v_H > 0$, and the lower one is stable. In this case, pick any $v_A \in (\underline{v}, \bar{v})$, where $\underline{v}$ is the maximum of 0, another SME to the left of v_L if it exists, and the horizontal intercept of Ψ. Now pick any $v_B \in (v_L, \bar{v})$ where $\bar{v}$ is the minimum of the next SME to the right of v_L or the v corresponding to the upper bound $\bar{q}$. From the graph, clearly, $\Psi(v_B) > v_B > v_A > \Psi(v_A)$, which is easily shown to imply that the ε's given in (23) are in $(0,1)$. The right panel shows the case where the higher SME v_H is stable, which is similar. ■

B. It is well known that with nonconvexities, including indivisible assets, it may be desirable to trade using lotteries. Without loss in generality we restrict attention to the case where sellers deliver the goods with probability 1 while buyers hand over their assets with probability $\pi \leq 1$, and it is possible to have $q < q^*$ and $\pi = 1$, or $q = q^*$ and $\pi < 1$, but not $q < q^*$ and $\pi < 1$ or $q > q^*$.

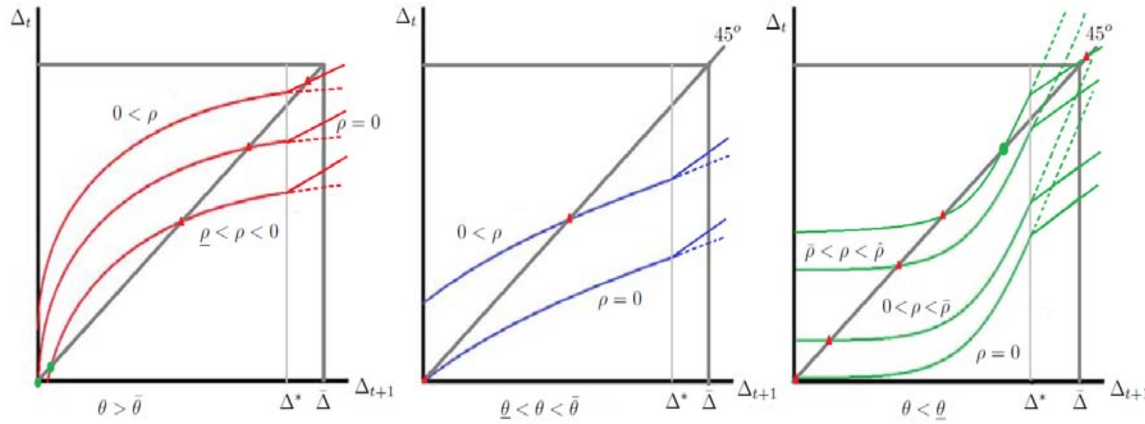

Figure 7: DME with lotteries

With lotteries, for any π_t we have

$$V_{1t} = \alpha_1 \left[u(q_t) - \beta \pi_t \Delta_{t+1} \right] + \beta V_{1t+1} + \rho \tag{24}$$

$$V_{0t} = \alpha_0 \left[\pi_t \beta \Delta_{t+1} - c(q_t) \right] + \beta V_{0t+1}, \tag{25}$$

Consider generalized Nash bargaining (other bargaining solutions and posting are similar),

$$\max_{q_t, \pi_t} \left[u(q_t) - \pi_t \beta \Delta_{t+1} \right]^{\theta} \left[\pi_t \beta \Delta_{t+1} - c(q_t) \right]^{1-\theta} \text{ st } \pi_t \leq 1,$$

given the constraints $q_t, \pi_t \geq 0$ are slack, as they must be if there are gains from trade. When $\beta \Delta_{t+1} < \theta c(q^*) + (1-\theta) u(q^*)$ the solution is $\pi = 1$ and q is the same as without lotteries; otherwise it is $q = q^*$ and

$$\pi_t = \frac{\theta c(q^*) + (1-\theta) u(q^*)}{\beta \Delta_{t+1}}. \tag{26}$$

A difference from the benchmark model is that now there is an equilibrium with trade at $q > 0$ no matter how big ρ gets: rather than hoarding the asset, when ρ is big, buyers use it to acquire $q = q^*$ by offering it to the seller with probability $\pi < 1$, with $\pi \to 0$ as $\rho \to \infty$. So asset circulation slows down as ρ rises but never stops.

For a general mechanism $v(q_t) = \pi_t \beta \Delta_{t+1}$, with lotteries the dynamical system is $\Delta_t = \Phi^*(\Delta_{t+1})$, where

$$\begin{aligned} \Phi^*(\Delta) \equiv\ & \alpha_1 u \circ Q^*(\beta\Delta) + \alpha_0 c \circ Q^*(\beta\Delta) + \rho \\ & + \left[1 - (\alpha_1 + \alpha_0)\pi^*(\beta\Delta)\right] \beta\Delta \end{aligned} \tag{27}$$

with $Q^*(\beta\Delta) = Q(\beta\Delta)$ and $\pi^*(\beta\Delta) = 1$ if $Q(\beta\Delta) < q^*$, while $Q^*(\beta\Delta) = q^*$ and $\pi(\beta\Delta) = v(q^*)/\beta\Delta^*$ otherwise. Thus Φ^* is linear for $\Delta_{t+1} > \Delta^* = v(q^*)/\beta$. Figure 7 amends Figure 2 to allow lotteries (dashed curves reproducing the case without lotteries). When there is two SME without lotteries, either the higher one is eliminated (as in the second highest curve in the right panel), or it survives and we introduce another SME with $\Delta > \Delta^*$ (as in the highest curve in the right panel). But the main point is that lotteries do not eliminate multiplicity.